MS-C1300 Complex Analysis

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This document is a skeleton of the theoretical content of the Fall 2024 course MS-C1300 Complex Analysis at Aalto University. The skeleton includes statements of results and their logical interdependencies, but no proofs, no examples, and no visualizations.

Instead, the hand-written lecture notes on the course home page correspond to the contents of the lectures. They include discussions of examples and informal ideas as well as the main theorems and their proofs. This theory skeleton is meant to complement the lectures as an organizational document: to indicate the flow of the theory development, and to clarify the main goals of the course and the purposes of the intermediate results.

An attempt is made to provide references to corresponding parts of the textbook: *An Introduction to Complex Function Theory* (Undergraduate Texts in Mathematics) by Bruce Palka.

This skeleton is work in progress. It is guaranteed to contain mistakes, the number and severity of which will be reduced with updates during the course. Please inform the lecturer Kalle Kytölä of errors that you notice!

Chapter 1

The complex number system

1.1 The field of complex numbers

Definition 1.1 (Complex numbers and their arithmetic operations [Palka1991, Sec. I.1.1])**.** The set of **complex numbers** is $\mathbb{C} = \mathbb{R} \times \mathbb{R}$, i.e., the set of pairs (x, y) of real numbers $x, y \in \mathbb{R}$.

The operations of addition and multiplication on $\mathbb{C}$ are defined by the formulas

$$(x_1, y_2) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2).$$

Denote $0 = (0, 0) \in \mathbb{C}$ and $1 = (1, 0) \in \mathbb{C}$.

For $z = (x, y) \in \mathbb{C}$, denote $-z = (-x, -y) \in \mathbb{C}$ and if $z \neq 0$ then denote $z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \in \mathbb{C}$.

We write a complex number (x, y) as $x + \mathfrak{i}\, y$. The compex number $\mathfrak{i} = (0, 1) \in \mathbb{C}$ is called the **imaginary unit**.

Typically used variable names for complex number are $z, w, \zeta \in \mathbb{C}$ etc.

Theorem 1.2 (The field of complex numbers [Palka1991, Sec. I.1.1])**.**

The set $\mathbb{C}$ of compex numbers with its operations of addition and multiplication, is a ***field****, i.e., the following properties hold for all $z, w, z_1, z_2, z_3 \in \mathbb{C}$:*

- $z + w = w + z$ ***(commutativity of addition)***
- $zw = wz$ ***(commutativity of multiplication)***
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ ***(associativity of addition)***
- $z_1(z_2z_3) = (z_1z_2)z_3$ ***(associativity of multiplication)***
- $0 = 0 + 0\,\mathfrak{i} = (0, 0) \in \mathbb{C}$ *satisfies* $z + 0 = z$ ***(neutral element for addition)***
- $1 = 1 + 0\,\mathfrak{i} = (1, 0) \in \mathbb{C}$ *satisfies* $z \cdot 1 = z$ ***(neutral element for multiplication)***
- $z + (-z) = 0$ *for any* $z \in \mathbb{C}$ ***(opposite element / additive inverse)***
- $z\, z^{-1} = 1$ *for any* $z \in \mathbb{C} \setminus \{0\}$ ***(multiplicative inverse)***
- $(z_1 + z_2)w = z_1w + z_2w$ ***(distributivity)****.*

Proof. Straightforward calculations using the definitions of the operations (Definition 1.1). □

1.2 Conjugate, modulus, and argument

Definition 1.3 (Complex conjugate [Palka1991, Sec. I.1.2])**.** The **complex conjugate** of a complex number $z = x + \mathfrak{i}y$ (where $x, y \in \mathbb{R}$) is the complex number $\overline{z} = x - \mathfrak{i}y$.

Lemma 1.4 (Properties of complex conjugate [Palka1991, Sec. I.1.2 (1.1)])**.** *For any $z, w \in \mathbb{C}$, we have*

$$\overline{\overline{z}} = z, \qquad \overline{z + w} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{z}\,\overline{w},$$

$$\mathfrak{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \mathfrak{Im}(z) = \frac{z - \overline{z}}{2\mathfrak{i}}.$$

Proof. Direct calculations. □

Definition 1.5 (Absolute value (modulus) [Palka1991, Sec. I.1.2])**.** The **absolute value** (or **modulus**) of a complex number $z = x + \mathfrak{i}y$ (where $x, y \in \mathbb{R}$) is the nonnegative real number $|z| = \sqrt{x^2 + y^2} \geq 0$.

Lemma 1.6 (Properties of absolute value [Palka1991, Sec. I.1.2 (1.2)])**.** *For any $z, w \in \mathbb{C}$, we have*

$$|z|^2 = z\,\overline{z}, \qquad |zw| = |z|\,|w|,$$

$$\mathfrak{Re}(z) \leq |z|, \qquad \mathfrak{Im}(z) \leq |z|,$$

$$|z + w| \leq |z| + |w|, \qquad |z + w| \geq \big||z| - |w|\big|.$$

Also, if $z \neq 0$, then

$$z^{-1} = \frac{\overline{z}}{|z|^2}, \qquad \Big|\frac{w}{z}\Big| = \frac{|w|}{|z|}.$$

Proof. Straightforward. □

Definition 1.7 (Argument [Palka1991, Sec. I.1.2])**.** A real number $\theta \in \mathbb{R}$ is an **argument** of a complex number $z \in \mathbb{C}$ if

$$z = |z|\,\Big(\cos(\theta) + \mathfrak{i}\,\sin(\theta)\Big).$$

(Note/warning: For a nonzero complex number z, it is convenient to denote $\theta = \arg(z)$, but this is an abuse of notation, the argumentis defined only modulo addition of integer multiples of 2π.)

The **principal argument** of a nonzero complex number $z \in \mathbb{C}$ is its unique argument on the interval $(-\pi, \pi]$, and it is denoted by $\mathrm{Arg}(z)$.

Lemma 1.8 (Discontinuity of the principal argument)**.** *The principal argument* $\mathrm{Arg} \colon \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ *is continuous on the subset* $\mathbb{C} \setminus (-\infty, 0]$*, but it is discontinuous on the negative real axis* $(-\infty, 0]$.

1.3 The polar form

Definition 1.9 (Complex exponential function)**.** We define the **complex exponential function** $\exp : \mathbb{C} \to \mathbb{C}$ by

$$\exp(x + \mathrm{i}y) = e^x \big(\cos(y) + \mathrm{i}\, \sin(y) \big) \qquad \text{for } x, y \in \mathbb{R},$$

where e^x is the usual real exponential. We also use the notation $e^z = \exp(z)$ for complex exponentials.

The exponential with purely imaginary argument takes the form of **Euler's formula**

$$e^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\, \sin(\theta) \qquad \text{for } \theta \in \mathbb{R}.$$

Lemma 1.10 (Properties of the complex exponential)**.** *For any $z, w \in \mathbb{C}$ we have*

$$e^{z+w} = e^z\, e^w.$$

For any $z \in \mathbb{C}$ we have

$$e^{\overline{z}} = \overline{e^z}, \quad |e^z| = e^{\mathfrak{Re}(z)}, \quad \arg(e^z) = \mathfrak{Im}(z) \pmod{2\pi}.$$

For $z, w \in \mathbb{C}$ we have $e^z = e^w$ if and only if $z = w + 2\pi\mathrm{i}n$ for some $n \in \mathbb{Z}$.

Proof. ... □

Lemma 1.11 (Polar form)**.** *Every complex number $z \in \mathbb{C}$ can be written in the* ***polar form***

$$z = r\, e^{\mathrm{i}\theta} \qquad \text{where } r \geq 0 \text{ and } \theta \in \mathbb{R}.$$

The modulus of z is the number $r = |z|$ above. If $z \neq 0$, then θ above is a choice of the argument of z, i.e., $\theta = \mathrm{Arg}(z) + 2\pi m$ for some $m \in \mathbb{Z}$.

Proof. ... □

Lemma 1.12 (Multiplication in polar form [Palka1991, Sec. I.1.2 (1.6)])**.** *For any $z, w \in \mathbb{C}$, written in polar form as $z = re^{\mathrm{i}\theta}$ and $w = r'e^{\mathrm{i}\theta'}$, the product can be written in polar form as*

$$zw = rr'\, e^{\mathrm{i}(\theta+\theta')}.$$

In other words,

$$|zw| = |z|\,|w| \qquad \text{and} \qquad \arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}.$$

Proof. ... □

Theorem 1.13 (De Moivre's formula [Palka1991, Sec. I.1.2 (1.7)])**.** *For any $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have*

$$\big(\cos(\theta) + \mathrm{i}\, \sin(\theta) \big)^n = \cos(n\theta) + \mathrm{i}\, \sin(n\theta).$$

Proof. Induction using Lemma 1.12. □

Lemma 1.14 (Roots of unity)**.** *For any $n \in \mathbb{N}$, the solutions $z \in \mathbb{C}$ to the equation*

$$z^n = 1$$

are the n distinct complex numbers

$$z_j \;=\; e^{\mathrm{i}2\pi j/n} \;=\; \cos\left(\frac{2\pi j}{n}\right) + \mathrm{i}\,\sin\left(\frac{2\pi j}{n}\right) \qquad \text{where } j = 0, 1, \ldots, n-1.$$

*These solutions are called the (complex) nth **roots of unity**.*

In particular, we have the polynomial factorization

$$z^n - 1 \;=\; \prod_{j=0}^{n-1} \left(z - e^{\mathrm{i}2\pi j/n}\right).$$

Proof. ... □

1.4 Functions of a complex variable

1.4.1 Polynomials and rational functions

Definition 1.15 (Polynomial)**.** Polynomial functions are functions $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{C}$ are coefficients.

Definition 1.16 (Rational function)**.** Rational functions are functions $f : D \to \mathbb{C}$ which can be written as ratios $f(z) = \frac{p(z)}{q(z)}$ of two polynomials $p, q \colon \mathbb{C} \to \mathbb{C}$ on a domain $D \subset \mathbb{C}$ where the denominator polynomial q has no zeroes.

1.4.2 Exponentials and branches of logarithms

Definition 1.17 (Principal complex logarithm)**.** The **principal logarithm** is the function

$$\begin{aligned} \mathrm{Log}\colon\; \mathbb{C} \setminus \{0\} &\to \mathbb{C} \\ \mathrm{Log}(z) \;&=\; \log|z| + \mathrm{i}\,\mathrm{Arg}(z), \end{aligned}$$

where $\log|z|$ is the usual natural logarithm of the positive real number $|z| > 0$ and $\mathrm{Arg}(z) \in (-\pi, \pi]$ is the principal argument of the nonzero complex number $z \neq 0$.

(Directly from this definition one sees that for $z \in \mathbb{C} \setminus \{0\}$ we have $e^{\mathrm{Log}(z)} = z$. All complex solutions w to $e^w = z$ are of the form $w = \mathrm{Log}(z) + 2\pi\mathrm{i}n$ where $n \in \mathbb{Z}$.)

Definition 1.18 (Branches of complex logarithm)**.** A **branch of the logarithm** is a continuous function $\ell\colon U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ such that

$$e^{\ell(z)} = z \qquad \text{for all } z \in U.$$

For example, the principal logarithm Log restricted to the open set $\mathbb{C} \setminus (-\infty, 0]$ is called the principal branch of the logarithm. Note that this principal branch cannot be extended continuously to the negative real axis.

(Note that since $e^w \neq 0$ for all $w \in \mathbb{C}$, any branch of the logarithm must exclude the origin from its domain of definition, $0 \notin U$.)

1.4.3 Complex power functions

Definition 1.19 (Principal complex power function)**.** Let $\alpha \in \mathbb{C}$. The **principal (complex) αth power function** is the function $\mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by

$$z \mapsto z^{\alpha} := e^{\alpha \operatorname{Log}(z)}.$$

(Note: Integer powers have more direct natural definitions. For $n \in \mathbb{N}$ we simply define z^n by recursive multiplication and the function $z \mapsto z^n$ is continuous and defined in all of $\mathbb{C}$ and coincides with the principal power function with $\alpha = n$ on $\mathbb{C} \setminus \{0\}$. We also define z^{-n} by recursive multiplication of the inverse z^{-1} of z, and the function $z \mapsto z^{-n}$ is continuous on $\mathbb{C} \setminus \{0\}$ and coincides with the principal power function with $\alpha = -n$. For $n = 0$ we define $z^0 = 1$ for any $z \in \mathbb{C}$, and this coincides with the principal power function with $\alpha = 0$ on $\mathbb{C} \setminus \{0\}$.)

Given a branch $\ell \colon U \to \mathbb{C}$ of logarithm on an open set $U \subset \mathbb{C}$, we obtain a branch of the αth power function on U by the formula $z \mapsto e^{\alpha \ell(z)}$. Using the same branch of the logarithm for the power functions, we have $z^{\alpha} z^{\beta} = z^{\alpha+\beta}$.

1.4.4 Branches of nth roots

Definition 1.20 (Principal nth root function)**.** Let $n \in \mathbb{N}$. The **principal (complex) nth root** of $z \in \mathbb{C} \setminus \{0\}$ is

$$\sqrt[n]{z} := z^{1/n} = e^{\frac{1}{n} \operatorname{Log}(z)}.$$

(It follows directly from the definition and the properties of complex exponential that $(\sqrt[n]{z})^n = z$. All complex solutions w to $w^n = z$ are of the form $w = \zeta \sqrt[n]{z}$ where $\zeta = e^{2\pi i j/n}$ with $j = 0, 1, \dots, n-1$, i.e., ζ is one of the n complex nth roots of unity.)

Chapter 2

Complex derivatives and analytic functions

2.1 Real linear maps versus complex linear maps

The right abstract way of understanding the differential of a function is as a linear approximation to a function locally. The key difference between real analysis and complex analysis is whether one uses real linear maps or complex linear maps.

Definition 2.1 (Linear map)**.** Let $\mathbb{K}$ be a field (for example $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), and let V and W be vector spaces over $\mathbb{K}$. A function $L : V \to W$ is said to be $\mathbb{K}$**-linear** if

$$\begin{aligned} L(v_1 + v_2) &= L(v_1) + L(v_2) && \text{for all } v_1, v_2 \in V, \\ L(cv) &= c\,L(v) && \text{for all } v \in V, c \in \mathbb{K}. \end{aligned}$$

Such a function L is also called a $\mathbb{K}$**-linear map** (or a $\mathbb{K}$**-linear transformation**) between the spaces V and W.

The complex plane $\mathbb{C} \cong \mathbb{R}^2$ can be seen either as a 2-dimensional real vector space or as a 1-dimensional complex vector space. In particular, it makes sense to talk about both $\mathbb{R}$-linear maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C}$-linear maps $\mathbb{C} \to \mathbb{C}$.

More generally, any complex vector space can be seen as a real vector space (of twice the same dimension), and any complex linear map becomes a real linear map. The converse is not true! Let us elaborate on this in a simple example which will soon be seen to pertain to the difference of complex differentiability and real differentiability.

Remark: Identifying $\mathbb{C} = \mathbb{R}^2$ (and choosing basis vectors $1, \mathfrak{i} \in \mathbb{C}$ for $\mathbb{C}$ seen as a 2-dimensional vector space), we see that an $\mathbb{R}$-linear map $L : \mathbb{C} \to \mathbb{C}$ can be encoded in a 2×2 matrix with real entries,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

in such a way that

$$L(x + \mathfrak{i}y) = (ax + by) + \mathfrak{i}(cx + dy).$$

Remark: A $\mathbb{C}$-linear map $L : \mathbb{C} \to \mathbb{C}$ can be encoded in a single complex number $\lambda \in \mathbb{C}$ (or more pedantically, in a 1×1 matrix $[\lambda] \in \mathbb{C}^{1\times 1}$), in such a way that

$$Lz = \lambda z.$$

We can write $\lambda = \alpha + \mathfrak{i}\beta$, with $\alpha = \mathfrak{Re}(\lambda), \beta = \mathfrak{Im}(\lambda) \in \mathbb{R}$. We can also write $z = x + \mathfrak{i}y$ and obtain the expression

$$L(x + \mathfrak{i}y) = (\alpha + \mathfrak{i}\beta)(x + \mathfrak{i}y) = (\alpha x - \beta y) + \mathfrak{i}(\beta x + \alpha y).$$

In other words, seen as a real-linear map, the complex multiplication by λ corresponds to the matrix

$$M = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

This clearly shows that not every real-linear map $\mathbb{C} \to \mathbb{C}$ is complex linear. It also gives an explicit set of equations for the entries of the matrix of a real-linear map characterizing complex-linearity, which turn out to be very closely related to the Cauchy-Riemann equations.

Lemma 2.2 (Complex linear versus real linear maps of $\mathbb{C}$)**.** *Let $L : \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$-linear map represented in the basis $1, \mathfrak{i}$ by the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Then the following are equivalent:*

- *L is $\mathbb{C}$-linear;*
- *$b = -c$ and $a = d$.*

Proof. Clear from the above discussion. □

2.2 Complex derivative

Definition 2.3 (Complex derivative [Palka1991, Sec. III.1.1])**.** Let $f \colon A \to \mathbb{C}$ be a complex-valued function defined on a subset $A \subset \mathbb{C}$ of the complex plane, and let $z_0 \in A$ be an interior point of the subset.

The f is said to have a **complex derivative**

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at z_0, if the limit on the right hand side above exists.

(In complex analysis we often drop the epithet "complex" above, and simply call $f'(z_0)$ the **derivative** of f at z_0.)

Lemma 2.4 (Local linear approximation)**.** *If a function $f \colon A \to \mathbb{C}$ has complex derivative $f'(z_0) = \lambda \in \mathbb{C}$ at a point $z_0 \in A$, then we can write a linear approximation*

$$f(z) = f(z_0) + (z - z_0)\,\lambda + \epsilon(z),$$

where the error term ϵ is small near z_0 in the sense that $\lim_{z \to z_0} \frac{\epsilon(z)}{|z - z_0|} = 0$.

Proof. ... □

Lemma 2.5 (Complex differentiability implies continuity [Palka1991, Sec. III.1.1])**.** *If a function $f \colon A \to \mathbb{C}$ has a complex derivative $f'(z_0)$ at a point $z_0 \in A$, then it is continuous at z_0.*

Proof. ... □

2.3 Cauchy-Riemann equations

Lemma 2.6 (Complex derivative implies differentiability)**.** *Let $f\colon A \to \mathbb{C}$ be a function defined on a set $A \subset \mathbb{C}$, and let $u\colon A \to \mathbb{R}$ and $v\colon A \to \mathbb{R}$ be its real and imaginary parts, viewed as real-valued functions of two real variables, $u(x,y) = \mathfrak{Re}(f(x+\mathrm{i}y))$ and $v(x,y) = \mathfrak{Im}(f(x+\mathrm{i}y))$, so that $f = u + \mathrm{i}\, v$. If f has a complex derivative $f'(z_0)$ at an interior point $z_0 = x_0 + \mathrm{i}y_0 \in A$, then u and v are differentiable at (x_0, y_0) and their partial derivatives satisfy the **Cauchy-Riemann equations***

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

(These equations are equivalent to the differential $\mathrm{d}f(x_0,y_0)\colon \mathbb{R}^2 \to \mathbb{R}^2$ being $\mathbb{C}$-linear when we identify $\mathbb{R}^2 = \mathbb{C}$.)

We can then write the derivative at z_0 in any of the following ways:

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0,y_0) + \mathrm{i}\,\frac{\partial v}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) - \mathrm{i}\,\frac{\partial u}{\partial y}(x_0,y_0) \\ &= \frac{\partial u}{\partial x}(x_0,y_0) - \mathrm{i}\,\frac{\partial u}{\partial y}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) + \mathrm{i}\,\frac{\partial v}{\partial x}(x_0,y_0). \end{aligned}$$

Proof. ... □

2.3.1 Differentiation rules

Lemma 2.7 (Linearity of the derivative [Palka1991, Sec. III.1.2 (3.4)])**.** *If two functions $f, g\colon A \to \mathbb{C}$ have complex derivatives $f'(z_0), g'(z_0)$ at a point $z_0 \in A$, then the sum function $f+g$ has a complex derivative at z_0 given by*

$$(f+g)'(z_0) = f'(z_0) + g'(z_0).$$

If a function $f\colon A \to \mathbb{C}$ is has a complex derivative $f'(z_0)$ at a point $z_0 \in A$ and $c \in \mathbb{C}$ is a complex number, then the function cf has complex derivative

$$(cf)'(z_0) = c\, f'(z_0)$$

at z_0.

Proof. ... □

Lemma 2.8 (Leibniz rule [Palka1991, Sec. III.1.2 (3.4)])**.** *If two functions $f, g\colon A \to \mathbb{C}$ have complex derivatives $f'(z_0), g'(z_0)$ at a point $z_0 \in A$, then the product function fg has complex derivative*

$$(fg)'(z_0) = f'(z_0)\, g(z_0) + f(z_0)\, g'(z_0)$$

at z_0.

Proof. ... □

Lemma 2.9 (Derivative of a quotient [Palka1991, Sec. III.1.2 (3.4)])**.** *If two functions $f, g\colon A \to \mathbb{C}$ have complex derivatives $f'(z_0), g'(z_0)$ at a point $z_0 \in A$ and $g(z_0) \neq 0$, then the quotient function f/g has complex derivative*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\, g(z_0) - f(z_0)\, g'(z_0)}{g(z_0)^2}.$$

at z_0.

Proof. ... □

Lemma 2.10 (Chain rule [Palka1991, Thm III.1.1])**.** *If $f\colon A \to B \subset \mathbb{C}$ is differentiable at $z_0 \in A$ and $g\colon B \to \mathbb{C}$ is differentiable at $f(z_0) \in B$, then the composition $g \circ f\colon A \to \mathbb{C}$ is differentiable at z_0, with derivative*

$$(g \circ f)'(z_0) = f'(z_0)\, g'(f(z_0)).$$

Proof. ... □

Lemma 2.11 (Derivative of inverse [Palka1991, Thm III.4.1])**.** *Suppose that f is a complex-valued function defined on a subset of the complex plane, which has a nonzero complex derivative $f'(z_0) \neq 0$ at a point z_0 and which has a local inverse function near z_0 in the sense that there are open sets $U, V \subset \mathbb{C}$ with $z_0 \in U$ and $f(z_0) \in V$, and the restriction of f to U is continuous $U \to V$ with a continuous inverse. Then the local inverse function $f^{-1}\colon V \to U$ has complex derivative at $w_0 := f(z_0)$ given by*

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}.$$

Proof. ... □

2.3.2 Analytic functions

Definition 2.12 (Analytic function [Palka1991, Sec. III.1.3])**.** A function $f\colon U \to \mathbb{C}$ defined on an open set $U \subset \mathbb{C}$ is said to be **analytic** (or **holomorphic**) if it is complex differentiable at every point $z_0 \in U$.

Theorem 2.13 (Cauchy-Riemann equations [Palka1991, Thm III.2.2])**.** *Let $f\colon U \to \mathbb{C}$ be a function defined on an open set $U \subset \mathbb{C}$, and let $u\colon U \to \mathbb{R}$ and $v\colon U \to \mathbb{R}$ be its real and imaginary parts, viewed as real-valued functions of two real variables,*

$$u(x,y) = \mathfrak{Re}\Big(f(x + \mathrm{i}y)\Big) \quad \text{and} \quad v(x,y) = \mathfrak{Im}\Big(f(x + \mathrm{i}y)\Big)$$

so that $f = u + \mathrm{i}\, v$.

Then the following are equivalent:

- *The functions u and v are differentiable at every point in U and their partial derivatives satisfy the **Cauchy-Riemann equations***

  $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

  in U.

- *The function f is analytic.*

Proof. ... □

Lemma 2.14 (Analytic functions are continuous)**.** *Every function $f\colon U \to \mathbb{C}$ which is analytic on an open set $U \subset \mathbb{C}$ is also continuous on U.*

Proof. ... □

Lemma 2.15 (Polynomials are analytic)**.** *Every polynomial function $p\colon \mathbb{C} \to \mathbb{C}$ is analytic.*

Proof. ... $\square$

Lemma 2.16 (Rational functions are analytic)**.** *Every rational function $f\colon U \to \mathbb{C}$ is analytic on its domain of definition $U \subset \mathbb{C}$.*

Proof. ... $\square$

Lemma 2.17 (The complex exponential is analytic)**.** *The complex exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ *is analytic. Its (complex) derivative at $z \in \mathbb{C}$ is $\exp'(z) = \exp(z)$.*

Proof. ... $\square$

Lemma 2.18 (Branches of nth root functions are analytic)**.** *The principal branch of the nth root function $z \mapsto \sqrt[n]{z}$ is analytic on its domain $\mathbb{C} \setminus (-\infty, 0]$.*

(Different branch choices can be made to obtain analyticity on other domains, but for $n \geq 2$, no branch of $\sqrt[n]{z}$ can be made analytic on all of $\mathbb{C}$.)

Proof. ... $\square$

2.3.3 Consequences of Cauchy-Riemann equations

Lemma 2.19 (Analytic functions of vanishing derivative)**.** *Suppose that $f\colon D \to \mathbb{C}$ is a analytic function on a connected open subset $D \subset \mathbb{C}$ of the complex plane such that $f'(z) = 0$ for all $z \in D$. Then f is a constant function.*

Proof. ... $\square$

Theorem 2.20 (Criteria for constantness of a analytic function)**.** *Suppose that $f\colon D \to \mathbb{C}$ is a analytic function on a connected open subset $D \subset \mathbb{C}$ of the complex plane. If any of the functions $u = \mathfrak{Re}(f)\colon D \to \mathbb{R}$, $v = \mathfrak{Im}(f)\colon D \to \mathbb{R}$, $|f|\colon D \to \mathbb{R}$, is constant on D, then f is itself a constant function.*

Proof. ... $\square$

Lemma 2.21 (Harmonicity of real and imaginary parts)**.** *Suppose that $f\colon U \to \mathbb{C}$ is a analytic function on an open subset $U \subset \mathbb{C}$ of the complex plane. Let $u, v\colon U \to \mathbb{R}$ denote the real and imaginary parts of f defined by $u(x, y) = \mathfrak{Re}(f(x + \mathfrak{i}y))$ and $v(x, y) = \mathfrak{Im}(f(x + \mathfrak{i}y))$. Assume moreover that that u and v are twice continuously differentiable (later it will be shown that this assumption holds automatically by the analyticity of f). Then u and v are **harmonic functions**, i.e., they satisfy*

$$\triangle u = 0 \qquad \text{and} \qquad \triangle v = 0, \qquad \text{where} \qquad \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. ... $\square$

Definition 2.22 (Harmonic conjugate)**.** Suppose that $u\colon U \to \mathbb{R}$ is harmonic function on an open subset $U \subset \mathbb{R}^2$, i.e., a twice continuously differentiable function satisfying $\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$ on U. A function $v\colon U \to \mathbb{R}$ is called a **harmonic conjugate** of u if the function

$$x + \mathfrak{i}y \ \mapsto \ u(x, y) + \mathfrak{i}\, v(x, y)$$

is analytic on U.

Lemma 2.23 (Local existence of harmonic conjugates)**.** *Let $B = \mathcal{B}(z_0; r) \subset \mathbb{C}$ be a disk in the complex plane. Suppose that $u\colon B \to \mathbb{R}$ is harmonic function on B. Then a harmonic conjugate $v\colon B \to \mathbb{R}$ of u in the disk B exists and is unique up to an additive constant.*

Proof. ... $\square$

Chapter 3

Contour integration

3.1 Complex-valued integrals

Definition 3.1 (Integral of a complex-valued function)**.** Let $f \colon [a,b] \to \mathbb{C}$ be a complex-valued continuous function defined on a closed interval $[a,b] \subset \mathbb{R}$. We define the **integral** of f as

$$\int_a^b f(t)\,\mathrm{d}t \;=\; \int_a^b \mathfrak{Re}\big(f(t)\big)\,\mathrm{d}t \;+\; \mathfrak{i}\int_a^b \mathfrak{Im}\big(f(t)\big)\,\mathrm{d}t.$$

(Note that on the right hand side we have just Riemann integrals of the continuous real-valued functions $t \mapsto \mathfrak{Re}\big(f(t)\big)$ and $t \mapsto \mathfrak{Im}\big(f(t)\big)$.)

Lemma 3.2 (Complex linearity of complex-valued integrals)**.** *If $f, g \colon [a,b] \to \mathbb{C}$ are complex-valued continuous functions defined on a closed interval $[a,b] \subset \mathbb{R}$, then the integral of their sum is*

$$\int_a^b \big(f(t) + g(t)\big)\,\mathrm{d}t \;=\; \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t.$$

If $f \colon [a,b] \to \mathbb{C}$ is a complex-valued continuous function defined on a closed interval $[a,b] \subset \mathbb{R}$, and $\lambda \in \mathbb{C}$ is a complex number, then the integral of the scalar multiple of f is

$$\int_a^b \lambda\, f(t)\,\mathrm{d}t \;=\; \lambda \int_a^b f(t)\,\mathrm{d}t.$$

Proof. ... □

Lemma 3.3 (Fundamental theorem of calculus for complex-valued integrals)**.** *Suppose that $f \colon [a,b] \to \mathbb{C}$ is a continuously differentiable complex-valued function on a closed interval $[a,b] \subset \mathbb{R}$. Denote its derivative by $\dot{f}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$. Then for the integral of the derivative of f we have*

$$\int_a^b \dot{f}(t)\,\mathrm{d}t \;=\; f(b) - f(a).$$

Proof. ... □

3.2 Paths in the complex plane

Definition 3.4 (Path [Palka1991, Sec. IV.1.1])**.** A **path** in the complex plane is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$ from a closed interval $[a,b] \subset \mathbb{R}$ to $\mathbb{C}$.

When $A \subset \mathbb{C}$ is a subset of the complex plane, we say that γ is a path in A if $\gamma(t) \in A$ for all $t \in [a,b]$.

If the starting point and the end point of the path γ are the same, $\gamma(a) = \gamma(b)$, then we say that γ is a **closed path**.

We sometimes want to disregard the parametrization, and view a path $\gamma\colon [a,b] \to \mathbb{C}$ as a subset of the complex plane. This subset is the range $\{\gamma(t) \,|\, t \in [a,b]\} \subset \mathbb{C}$ of the parametrizing function, but with a slight abuse of notation we often just write $\gamma \subset \mathbb{C}$ also for this subset.

(Note that the subset $\gamma \subset \mathbb{C}$ is compact, by continuity of $\gamma\colon [a,b] \to \mathbb{C}$ and compactness of $[a,b]$.)

Definition 3.5 (Smooth path [Palka1991, Sec. IV.1.2])**.** A path $\gamma\colon [a,b] \to \mathbb{C}$ is **smooth** if it is continuously differentiable, i.e., the derivative

$$\dot{\gamma}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$$

with respect to the parameter t exists for all $t \in [a,b]$ (one-sided derivatives at the interval end points a and b), and defines a continuous complex-valued function $t \mapsto \dot{\gamma}(t)$ on $[a,b]$.

Definition 3.6 (Contour / piecewise smooth path [Palka1991, Sec. IV.1.2])**.** A **contour** (also called a **piecewise smooth path**) is a continuous function $\gamma\colon [a,b] \to \mathbb{C}$ such that for some finite subdivision $a = t_0 < t_1 < \ldots < t_n = b$, the restrictions $\gamma|_{[t_{j-1},t_j]}$ to the subintervals $[t_{j-1}, t_j] \subset [a,b]$ are smooth paths for each $j = 1, \ldots, n$.

If the starting point and the end point of the contour γ are the same, $\gamma(a) = \gamma(b)$, then we say that γ is a **closed contour**.

Definition 3.7 (Reverse path [Palka1991, Sec. IV.1.4])**.** Given a path $\gamma\colon [a,b] \to \mathbb{C}$, the **reverse path** $\overleftarrow{\gamma}\colon [a,b] \to \mathbb{C}$ is the path defined by

$$\overleftarrow{\gamma}(t) = \gamma(a+b-t) \qquad \text{for } t \in [a,b].$$

Definition 3.8 (Concatenation of paths [Palka1991, Sec. IV.1.4])**.** Given path $\gamma\colon [a,b] \to \mathbb{C}$ and $\eta\colon [c,d] \to \mathbb{C}$ with $\gamma(b) = \eta(c)$ (the starting point of η coincides with the end point of γ), the **concatenation** of γ and η is the path $\gamma \oplus \eta\colon [a, b+d-c] \to \mathbb{C}$ defined by

$$(\gamma \oplus \eta)(t) = \begin{cases} \gamma(t) & \text{for } t \in [a,b], \\ \eta(c+t-b)) & \text{for } t \in [b, b+d-c]. \end{cases}$$

(The slightly cumbersome formula in the second case is due to the fact that we need to attach the two parameter intervals of lengths $b-a$ and $d-c$ to each other, and we have, somewhat arbitrarily, chosen to glue them to form the interval $[a, b+d-c]$.)

Definition 3.9 (Reparametrization of paths [Palka1991, Sec. IV.1.5])**.** Given a path $\gamma\colon [a,b] \to \mathbb{C}$ and a continuous increasing bijection $\phi\colon [c,d] \to [a,b]$, we define the **reparametrization** of γ by ϕ as the path

$$\begin{aligned} \gamma \circ \phi : \; [c,d] &\to \mathbb{C} \\ t &\mapsto \gamma(\phi(t)). \end{aligned}$$

Note that

- $\phi^{-1}\colon [a,b] \to [c,d]$ is also a continuous increasing bijection (a continuous bijection from the compact $[a,b]$ is automatically a homeomorphism; see Lemma A.33) and reparametrization can be undone by rereparametrizing by ϕ^{-1};
- If both γ and the reparametrization function ϕ are smooth (continuously differentiable), then the reparametrized path $\gamma \circ \phi$ is also smooth;
- If both γ and the reparametrization function ϕ are piecewise smooth, then the reparametrized path $\gamma \circ \phi$ is also piecewise smooth, i.e., a contour.

3.3 Integrals along paths

Definition 3.10 (Contour integral along a smooth path [Palka1991, Sec. IV.2.1])**.** Let $f : A \to \mathbb{C}$ be a continuous function defined on a subset $A \subset \mathbb{C}$. Let $\gamma\colon [a,b] \to A$ be a smooth path in A. We define the **integral of f along γ** as

$$\int_\gamma f(z)\,dz \;=\; \int_a^b f(\gamma(t))\,\dot\gamma(t)\,\mathrm{d}t.$$

(Here $\dot\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$ denotes the derivative of the smooth path γ with respect to its parameter t.)

Sometimes it is appropriate to integrate functions with respect to the arc length in the following sense.

Definition 3.11 (Arc length integral along a smooth path [Palka1991, Sec. IV.2.1])**.** Let $f : A \to \mathbb{C}$ be a continuous function defined on a subset $A \subset \mathbb{C}$. Let $\gamma\colon [a,b] \to A$ be a smooth path in A. We define the **integral of f with respect to the arc length of γ** as

$$\int_\gamma f(z)\,|dz| \;=\; \int_a^b f(\gamma(t))\,|\dot\gamma(t)|\,\mathrm{d}t.$$

(Here $\dot\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \in \mathbb{C}$ denotes the derivative of the smooth path γ with respect to its parameter t, and $|\dot\gamma(t)| \geq 0$ denotes the absolute value of this derivative.)

In order to extend the definition of contour integrals to piecewise smooth paths, we note that the definition behaves additively under path concatenation.

Lemma 3.12 (Contour integrals and smooth path concatenation [Palka1991, Lem IV.2.1(iv)])**.** *If a smooth path γ in A is a concatenation of smooth paths $\eta_1, \ldots, \eta_n$, and $f : A \to \mathbb{C}$ is a continuous function defined on $A \subset \mathbb{C}$, then we have*

$$\int_\gamma f(z)\,\mathrm{d}z \;=\; \sum_{j=1}^n \int_{\eta_j} f(z)\,\mathrm{d}z$$

and

$$\int_\gamma f(z)\,|\mathrm{d}z| \;=\; \sum_{j=1}^n \int_{\eta_j} f(z)\,|\mathrm{d}z|.$$

Proof. ... □

By virtue of the above, the following gives a well-defined meaning to integrals along piecewise smooth paths.

Definition 3.13 (Contour integral [Palka1991, Sec. IV.2.1])**.** Let $f : A \to \mathbb{C}$ be a continuous function defined on a subset $A \subset \mathbb{C}$. Let $\gamma\colon [a,b] \to A$ be a piecewise smooth path in A, which is a concatenation of smooth paths $\eta_1, \dots, \eta_n$. We define the **integral of f along γ** as

$$\int_\gamma f(z)\,dz \;=\; \sum_{j=1}^{n} \int_{\eta_j} f(z)\,dz.$$

Definition 3.14 (Arc-length integral)**.** Let $f : A \to \mathbb{C}$ be a continuous function defined on a subset $A \subset \mathbb{C}$. Let $\gamma\colon [a,b] \to A$ be a piecewise smooth path (i.e., a contour) in A, which is a concatenation of smooth paths $\eta_1, \dots, \eta_n$. We define the **integral of f with respect to the arc length of γ** as

$$\int_\gamma f(z)\,|dz| \;=\; \sum_{j=1}^{n} \int_{\eta_j} f(z)\,|dz|.$$

Definition 3.15 (Length of a path or a contour)**.** Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth path (i.e., a contour) in $\mathbb{C}$. The **length** $\ell(\gamma)$ of γ is defined as

$$\ell(\gamma) = \int_\gamma |\mathrm{d}z|.$$

Lemma 3.16 (Reparametrization invariance of integrals [Palka1991, Lem IV.2.1(v)])**.** *Let γ be a piecewise smooth path in A, and let $\tilde{\gamma}$ be obtained from γ by an orientation-preserving reparametrization. Then for any continuous function $f : A \to \mathbb{C}$ we have*

$$\int_{\tilde{\gamma}} f(z)\,dz \;=\; \int_\gamma f(z)\,dz$$

and

$$\int_{\tilde{\gamma}} f(z)\,|dz| \;=\; \int_\gamma f(z)\,|dz|.$$

Lemma 3.17 (Contour integrals and path reversal [Palka1991, Lem IV.2.1(iii)])**.** *If $f : A \to \mathbb{C}$ is a continuous function defined on $A \subset \mathbb{C}$, and γ is a piecewise path in A, then for the contour integral and the arc length integral behave as follows under path reversal: then we have*

$$\int_{\overline{\gamma}} f(z)\,\mathrm{d}z \;=\; -\int_\gamma f(z)\,\mathrm{d}z$$

and

$$\int_{\overline{\gamma}} f(z)\,|\mathrm{d}z| \;=\; \int_\gamma f(z)\,|\mathrm{d}z|$$

Proof. ... ☐

Lemma 3.18 (Linearity of integrals [Palka1991, Lem IV.2.1(i-ii)])**.**
Let $A \subset \mathbb{C}$ be a subset of the complex plane let and $\gamma\colon [a,b] \to A$ be a contour in A.

If $f, g\colon A \to \mathbb{C}$ are continuous functions defined on A, then the contour integral and the arc length integral of their sum are

$$\begin{aligned}\int_\gamma (f(z) + g(z))\,\mathrm{d}z \;&=\; \int_\gamma f(z)\,\mathrm{d}z + \int_\gamma g(z)\,\mathrm{d}z \\ \int_\gamma (f(z) + g(z))\,|\mathrm{d}z| \;&=\; \int_\gamma f(z)\,|\mathrm{d}z| + \int_\gamma g(z)\,|\mathrm{d}z|.\end{aligned}$$

If $f\colon A \to \mathbb{C}$ is a complex-valued continuous function defined on A, and $\lambda \in \mathbb{C}$ is a complex number, then the contour integral and the arc length integral of the scalar multiple of f are

$$\begin{aligned}\int_\gamma \lambda\, f(z)\,\mathrm{d}z \;&= \lambda \int_\gamma f(z)\,\mathrm{d}z \\ \int_\gamma \lambda\, f(z)\,|\mathrm{d}z| \;&= \lambda \int_\gamma f(z)\,|\mathrm{d}z|.\end{aligned}$$

Proof. ... □

Lemma 3.19 (Triangle inequality for contour integrals [Palka1991, Lem IV.2.1(vi)])**.** *Let $f : A \to \mathbb{C}$ be a continuous function defined on $A \subset \mathbb{C}$, and let γ be a contour in A. Then we have*

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \;\le\; \int_\gamma |f(z)|\,|dz|.$$

Proof. ... □

Corollary 3.20 (An a priori bound for contour integrals)**.** *Let $f : A \to \mathbb{C}$ be a continuous function defined on $A \subset \mathbb{C}$, and let γ be a contour in A. Assume that $|f(z)| \le M$ for all points z on the contour γ. Then we have*

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \;\le\; M\,\ell(\gamma),$$

where $\ell(\gamma) = \int_\gamma |\mathrm{d}z|$ denotes the length of the contour γ.

Proof. ... □

The following slightly technical auxiliary result will only be used later (for winding number properties and for Cauchy's formula for the derivative). But since the result only requires contour integration, the natural logical place for it is here. Also, strictly speaking, we only need the cases $k = 1$ and $k = 2$ in this lemma; but including general $k \in \mathbb{N}$ gives the quickest route to Cauchy's formula for higher order derivatives.

Lemma 3.21 ([Palka1991, Lemma V.1.6])**.** *Let γ be a contour in $\mathbb{C}$, and let $h\colon \gamma \to \mathbb{C}$ be a continuous function on the contour (we slightly abuse the notation here to identify the contour as a subset $\gamma \subset \mathbb{C}$). Let $k \in \mathbb{N}$ be a positive integer. Define $H\colon \mathbb{C} \setminus \gamma \to \mathbb{C}$ by*

$$H(z) = \int_\gamma \frac{h(\zeta)}{(\zeta - z)^k}\,\mathrm{d}\zeta.$$

Then H is analytic on $\mathbb{C} \setminus \gamma$, and its derivative at $z \in \mathbb{C} \setminus \gamma$ is given by

$$H'(z) = k \int_\gamma \frac{h(\zeta)}{(\zeta - z)^{k+1}}\,\mathrm{d}\zeta.$$

Proof. ... □

3.4 Primitives

Definition 3.22 (Primitive of a function [Palka1991, Sec. IV.2.3])**.** Let $f : U \to \mathbb{C}$ be a function defined on an open subset $U \subset \mathbb{C}$. A **primitive** of f is a function $F : U \to \mathbb{C}$ such that F is analytic (i.e., complex differentiable) on U,

$$F'(z) = f(z) \qquad \text{for all } z \in U.$$

Theorem 3.23 (Fundamental theorem of calculus for contour integrals [Palka1991, Thm IV.2.2])**.** *Suppose that $f\colon U \to \mathbb{C}$ is a continuous function on an open set $U \subset \mathbb{C}$, and that f has a primitive $F\colon U \to \mathbb{C}$. Then for any contour $\gamma\colon [a,b] \to U$ we have*

$$\int_\gamma f(z)\,dz \;=\; F(\gamma(b)) - F(\gamma(a)).$$

In particular for any closed contour γ in U, we have

$$\oint_\gamma f(z)\,dz \;=\; 0.$$

Proof. ... ☐

Lemma 3.24 (Existence of primitives for monomials)**.** *For $n \in \{0, 1, 2, ...\}$, the monomial function $f(z) = z^n$ has a primitive $F(z) = \frac{1}{n+1}z^{n+1} + c$ (with $c \in \mathbb{C}$ arbitrary) in the whole complex plane $\mathbb{C}$.*

For $n \in \{-2, -3, -4, ...\}$, the monomial function $f(z) = z^n$ has a primitive $F(z) = \frac{1}{n+1}z^{n+1} + c$ (with $c \in \mathbb{C}$ arbitrary) in the punctured complex plane $\mathbb{C} \setminus \{0\}$.

The monomial function $f(z) = z^{-1} = \frac{1}{z}$ does not have a primitive in the punctured complex plane $\mathbb{C} \setminus \{0\}$.

Proof. ... ☐

Theorem 3.25 (Characterization of the existence of primitives)**.** *Let $f\colon U \to \mathbb{C}$ be a continuous function on an open set $U \subset \mathbb{C}$. Then the following conditions are equivalent:*

(a) f has a primitive on U;

(b) the contour integrals $\int_\gamma f(z)\,\mathrm{d}z$ of f along contours γ in U only depend on the starting point and the end point of γ;

(c) for all closed contours γ in U we have $\oint_\gamma f(z)\,\mathrm{d}z = 0$.

Proof. ... ☐

Chapter 4

Cauchy's theorem and consequences

4.1 Convex, star-shaped, and simply connected domains

Definition 4.1 (Line segment)**.** Given two points $z_1, z_2 \in \mathbb{C}$ in the complex plane, the **line segment** between z_1 and z_2 is the path

$$\begin{aligned} \gamma\colon [0,1] &\to \mathbb{C} \\ \gamma(t) &= z_1 + t\,(z_2 - z_1). \end{aligned}$$

We also often view the line segment as a subset of $\mathbb{C}$ rather than a parametrized path, and we then denote it by

$$[z_1, z_2] \;=\; \Big\{ z_1 + t\,(z_2 - z_1) \;\Big|\; t \in [0,1] \Big\} \;\subset\; \mathbb{C}.$$

Definition 4.2 (Convex set)**.** A subset $A \subset \mathbb{C}$ of the complex plane is called **convex** if for any two points $z_1, z_2 \in A$, the line segment between them is contained in the subset,

$$[z_1, z_2] \;\subset\; A.$$

Definition 4.3 (Star-shaped set)**.** A subset $A \subset \mathbb{C}$ of the complex plane is called **star-shaped** if there exists a point $z_* \in A$ such that for any $z \in A$, the line segment between z_* and z is contained in the subset,

$$[z_*, z] \;\subset\; A.$$

Lemma 4.4 (Convex sets are star-shaped)**.** *Any nonempty convex set is star-shaped.*

Proof. ... □

Lemma 4.5 (Star-shaped sets are path connected and simply connected)**.** *Any star-shaped set $U \subset \mathbb{C}$ is path connected and simply connected.*

Proof. ... □

4.2 Cauchy's integral theorem

Lemma 4.6 (Goursat's lemma [Palka1991, Lem V.1.1])**.** *Suppose that a function $f\colon U \to \mathbb{C}$ is analytic on an open set $U \subset \mathbb{C}$. Then for any closed triangle $\triangle \subset U$, we have*

$$\oint_{\partial\triangle} f(z)\,\mathrm{d}z = 0.$$

Proof. ... □

Lemma 4.7 (Existence of primitives in star-shaped domains)**.** *Every analytic function $f\colon U \to \mathbb{C}$ on a star-shaped domain $U \subset \mathbb{C}$ has a primitive in U.*

Proof. ... □

Theorem 4.8 (Cauchy's integral theorem for star-shaped domains [Palka1991, Thm V.1.5])**.** *Suppose that a function $f\colon U \to \mathbb{C}$ is analytic on a star-shaped open subset $U \subset \mathbb{C}$. Then for any closed contour γ in U we have*

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0.$$

Proof. ... □

Corollary 4.9 (Local Cauchy's integral theorem [Palka1991, Thm V.5.1])**.** *Suppose that a function $f\colon U \to \mathbb{C}$ is analytic on a open set $U \subset \mathbb{C}$. Then for any disk $B \subset U$ contained in the domain U and any closed contour γ in B we have*

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 0.$$

Proof. ... □

4.3 Cauchy's integral formula

Theorem 4.10 (Cauchy's integral formula for star-shaped subdomains [Palka1991, Thm V.2.3])**.** *Suppose that a function $f\colon U \to \mathbb{C}$ is analytic on an open set $U \subset \mathbb{C}$, and suppose that γ is a closed contour in U parametrizing the boundary of a star-shaped Jordan subdomain $V \subset U$ in a counterclockwise orientation. Then for any point $z \in V$ we have*

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Proof. ... □

By far the most commonly used special case of Theorem 4.10 is when the contour γ is a circle, encircling a disk whose closure is contained in the domain of the analytic function (recall that disks are convex and therefore star-shaped).

Corollary 4.11 (Cauchy's integral formula for circles)**.** *Suppose that a function* $f\colon U \to \mathbb{C}$ *is analytic on an open set* $U \subset \mathbb{C}$*. Let* $\overline{\mathcal{B}}(z_0; r) \subset U$ *be a closed disk contained in* U*. Then for any point* $z \in \mathcal{B}(z_0; r)$ *we have*

$$f(z) = \frac{1}{2\pi\mathfrak{i}} \oint_{\partial\mathcal{B}(z_0;r)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

where the circle $\partial\mathcal{B}(z_0; r)$ *is parametrized in the counterclockwise orientation.*

Proof. ... □

4.4 Ideas underlying the generalizations

The generalizations of Cauchy's integral theorem and Cauchy's integral formula are based on the following homotopy invariance property of contour integrals (whose proof we do not do in detail in this course).

Lemma 4.12 (Homotopy invariance of contour integrals)**.** *Let* $f\colon U \to \mathbb{C}$ *be an analytic function on an open set* $U \subset \mathbb{C}$*, and let* γ_0 *and* γ_1 *be two closed contours in* U *which are homotopic to each other in* U*. Then we have*

$$\oint_{\gamma_0} f(z) \, \mathrm{d}z = \oint_{\gamma_1} f(z) \, \mathrm{d}z.$$

This readily implies the following generalization of Cauchy's integral theorem.

Theorem 4.13 (Cauchy's integral theorem [Palka1991, Thm V.5.1])**.** *Suppose that a function* $f\colon U \to \mathbb{C}$ *is analytic on a open set* $U \subset \mathbb{C}$*. Then for any contractible closed contour* γ *we have*

$$\oint_{\gamma} f(z) \, \mathrm{d}z = 0.$$

In particular, if U *is simply connected, then for any closed contour* γ *in* U *we have*

$$\oint_{\gamma} f(z) \, \mathrm{d}z = 0,$$

and the analytic function f *has a primitive in* U*.*

Proof. ... □

The other ingredient of generalization of Cauchy's integral formula to arbitrary contours and points not lying on those contours is the winding number of a contour around a point.

Definition 4.14 (Winding number)**.** Let $z \in \mathbb{C}$, and let γ be a closed contour in $\mathbb{C} \setminus \{z\}$. The **winding number** of γ around z is defined as

$$\mathfrak{n}_\gamma(z) = \frac{1}{2\pi\mathfrak{i}} \oint_{\gamma} \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Lemma 4.15 (Winding number concatenation and reversal)**.** *Let* γ *and* η *be closed contours in* $\mathbb{C}$ *both starting and ending at the same point* $z_0 \in \mathbb{C}$*. Then for any point* $z \in \mathbb{C} \setminus (\gamma \cup \eta)$ *we have*

$$\mathfrak{n}_{\gamma \oplus \eta}(z) \;=\; \mathfrak{n}_\gamma(z) + \mathfrak{n}_\eta(z)$$

and for any point $z \in \mathbb{C} \setminus \gamma$ *we have*

$$\mathfrak{n}_{\overline{\gamma}}(z) \;=\; -\mathfrak{n}_\gamma(z).$$

Proof. ... □

Lemma 4.16 (Winding number properties). *Let γ be a closed contour in $\mathbb{C}$. Then the winding numbers $\mathfrak{n}_\gamma(z)$ of points $z \in \mathbb{C} \setminus \gamma$ satisfy:*

(a) $z \mapsto \mathfrak{n}_\gamma(z)$ *is constant on each connected component of* $\mathbb{C} \setminus \gamma$*;*

(b) $\mathfrak{n}_\gamma(z) = 0$ *for all* z *in the unbounded connected component of* $\mathbb{C} \setminus \gamma$*;*

(c) If γ *is a Jordan contour and* $V \subset \mathbb{C} \setminus \gamma$ *is the bounded connected component of* $\mathbb{C} \setminus \gamma$*, then either* $\mathfrak{n}_\gamma(z) = 1$ *for all* $z \in V$ *or* $\mathfrak{n}_\gamma(z) = -1$ *for all* $z \in V$*.*

Proof. ... □

Lemma 4.17 (Homotopy invariance of winding numbers). *Let $z \in \mathbb{C}$ be a point and let γ and η be two closed contours in $\mathbb{C} \setminus \{z\}$ which are homotopic to each other in $\mathbb{C} \setminus \{z\}$. Then we have*

$$\mathfrak{n}_\gamma(z) = \mathfrak{n}_\eta(z).$$

Proof. ... □

The following is then a version of Cauchy's integral formula which has no restrictions on the closed contour and no restrictions on the position of the point with respect to the contour, except that the point must not lie on the contour (for otherwise there is a singularity in the integrand).

Theorem 4.18 (Cauchy's integral formula [Palka1991, Thm V.2.3]). *Suppose that a function $f \colon U \to \mathbb{C}$ is analytic on an open simply connected subset $U \subset \mathbb{C}$ of the complex plane. Then for any closed contour γ in U we have*

$$\oint_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \;=\; 2\pi\mathfrak{i}\, \mathfrak{n}_\gamma(z)\, f(z).$$

Proof. ... □

4.5 Analyticity of derivatives

Lemma 4.19 (Analyticity of derivatives [Palka1991, Thm V.3.1]). *If a function $f \colon U \to \mathbb{C}$ is analytic on an open set $U \subset \mathbb{C}$, then its derivative f' is also analytic on U. In particular, then f is continuously differentiable, $f \in \mathcal{C}^1(U)$.*

Proof. ... □

Corollary 4.20 (Analyticity of higher derivatives [Palka1991, Cor V.3.2]). *If a function $f \colon U \to \mathbb{C}$ is analytic on an open set $U \subset \mathbb{C}$, then its derivatives $f', f'', \ldots, f^{(k)}, \ldots$ of all orders are also analytic on U. In particular, then f is infinitely differentiable, $f \in \mathcal{C}^\infty(U)$.*

Proof. Straightforward induction using Lemma 4.19. □

Theorem 4.21 (Morera's theorem [Palka1991, Thm V.3.3]). *Let $f \colon U \to \mathbb{C}$ be a continuous function on an open set $U \subset \mathbb{C}$. If f has the property that*

$$\oint_{\partial\triangle} f(z)\, \mathrm{d}z = 0$$

for any closed triangle $\triangle \subset U$, then f is analytic on U.

Proof. ... □

Theorem 4.22 (Cauchy's integral formula for derivatives)**.** *Suppose that a function $f : U \to \mathbb{C}$ is analytic on an open simply connected subset $U \subset \mathbb{C}$ of the complex plane. Then for any closed contour γ in U, any $n \in \mathbb{N}$, and any point $z \in U$ we have*

$$\mathfrak{n}_\gamma(z)\, f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta.$$

Proof. ... □

Lemma 4.23 (Cauchy's estimate for derivatives [Palka1991, Thm V.3.6])**.** *Suppose that a function $f : U \to \mathbb{C}$ is analytic on an open set $U \subset \mathbb{C}$ containing the disk $\mathcal{B}(z_0; r) \subset U$, and suppose that there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathcal{B}(z_0; r)$. Then for any $n \in \mathbb{N}$ and any $z \in \mathcal{B}(z_0; r)$ we have the following bound for the nth derivative $f^{(n)}$ of f:*

$$\left| f^{(n)}(z) \right| \leq \frac{n!\, M\, r}{(r - |z - z_0|)^{n+1}}.$$

In particular, for the center point z_0 of the disk, we have

$$\left| f^{(n)}(z_0) \right| \leq n!\, M\, r^{-n}.$$

Proof. ... □

4.6 Liouville's theorem

Theorem 4.24 (Liouville's theorem [Palka1991, Thm V.3.7])**.** *If a function $f : \mathbb{C} \to \mathbb{C}$ on the entire complex plane is analytic and bounded, then f is a constant function.*

Proof. ... □

4.7 The fundamental theorem of algebra

Theorem 4.25 (Fundamental theorem of algebra [Palka1991, Thm V.3.8])**.** *Every non-constant polynomial function $p : \mathbb{C} \to \mathbb{C}$ has a root, i.e., there exists a $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. ... □

Corollary 4.26 (Polynomial factorization [Palka1991, Thm V.3.9])**.** *A complex-coefficient polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ of degree $n \in \mathbb{N}$ can be factored as*

$$p(z) = c\,(z - z_1)\,(z - z_2)\,\cdots\,(z - z_n)$$

where $c = a_n \neq 0$, and $z_1, \ldots, z_n \in \mathbb{C}$ are the roots of p (with repetition according to the multiplicities of the roots).

Proof. This follows from Theorem **??** by induction on the degree of the polynomial, using the polynomial division (Euclidean algorithm in the ring of univariate polynomials, see **MS-C1081 Abstract Algebra**). □

4.8 Maximum principle

Theorem 4.27 (Maximum principle for analytic functions [Palka1991, Thm V.3.10])**.** *Let $f\colon \mathcal{D} \to \mathbb{C}$ be an analytic function on a connected open set $\mathcal{D} \subset \mathbb{C}$. Suppose that there exists a point $z_0 \in \mathcal{D}$ such that*

$$|f(z)| \leq |f(z_0)| \qquad \text{for all } z \in \mathcal{D}.$$

Then f is a constant function.

Proof. ... □

Corollary 4.28 (Maximum principle for analytic functions continuous up to the boundary [Palka1991, Cor V.3.11])**.** *Let $\mathcal{D} \subset \mathbb{C}$ be a bounded connected open set. Let $f\colon \overline{\mathcal{D}} \to \mathbb{C}$ be a continuous function on its closure which is analytic in $\mathcal{D}$. Then $z \mapsto |f(z)|$ attains its maximum in $\overline{\mathcal{D}}$ at some point of the boundary $\partial\mathcal{D}$.*

Proof. ... □

Lemma 4.29 (Schwarz's lemma [Palka1991, Thm V.3.14])**.** *Let $f\colon \mathcal{B}(0;1) \to \mathbb{C}$ be an analytic function on the open unit disk such that $|f(z)| \leq 1$ for all $z \in \mathcal{B}(0;1)$ and $f(0) = 0$. Then we have*

$$|f'(0)| \leq 1 \qquad \text{and} \qquad |f(z)| \leq |z| \quad \text{for all } z \in \mathcal{B}(0;1).$$

Furthermore, unless f is of the form $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, then we have

$$|f'(0)| < 1 \qquad \text{and} \qquad |f(z)| < |z| \quad \text{for all } z \in \mathcal{B}(0;1) \setminus \{0\}\,.$$

Proof. ... □

4.9 The mean value property

Theorem 4.30 (Mean value property for analytic functions)**.** *Suppose that a function $f\colon U \to \mathbb{C}$ is analytic on an open set $U \subset \mathbb{C}$ containing the closed disk $\overline{\mathcal{B}}(z;r) \subset U$. Then we have*

$$f(z) = \frac{1}{2\pi r} \oint_{\partial\mathcal{B}(z;r)} \frac{f(\zeta)}{\zeta - z}\, |\mathrm{d}\zeta|.$$

Proof. ... □

Chapter 5

Power series

5.1 Uniform convergence

Definition 5.1 (Uniform convergence [Palka1991, Sec. VII.1.1])**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n \colon X \to \mathbb{C}$, and let $f \colon X \to \mathbb{C}$ also be a such function. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ **converges uniformly** to f (on X) if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\big|f_n(x) - f(x)\big| < \varepsilon \qquad \text{for all } x \in X.$$

Lemma 5.2 (Cauchy criterion for uniform convergence [Palka1991, Thm VII.1.2])**.** *Let $f_n \colon A \to \mathbb{C}$, $n \in \mathbb{N}$, be complex-valued functions defined on the same set A. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on A if and only if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$ and all $z \in A$ we have $|f_n(z) - f_m(z)| < \varepsilon$.*

*(When $(f_n)_{n\in\mathbb{N}}$ satisfies the condition above, it could be called a **uniform Cauchy sequence** on A.)*

Proof. ... □

Lemma 5.3 (Continuity is preserved in uniform limits [Palka1991, Thm VII.1.1])**.** *Let X be a metric space (e.g., $\mathbb{R}$, $\mathbb{C}$, or a subset of these). If a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n \colon X \to \mathbb{C}$ converges uniformly to a function $f \colon X \to \mathbb{C}$, then f is continuous.*

Proof. See **MS-C1541 Metric Spaces**. □

Lemma 5.4 (Integration commutes with uniform limits [Palka1991, Thm VII.1.1])**.** *If a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n \colon [a,b] \to \mathbb{C}$ on a closed interval $[a,b] \subset \mathbb{R}$ converges uniformly to a function $f \colon [a,b] \to \mathbb{C}$, then we have*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

Proof. ... □

Corollary 5.5 (Contour integration commutes with uniform limits [Palka1991, Thm VII.1.1])**.** *If a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n \colon A \to \mathbb{C}$ on a subset $A \subset \mathbb{C}$ of the complex plane converges uniformly to a function $f \colon A \to \mathbb{C}$, then for any piecewise smooth path γ in A we have*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Proof. This follows straightforwardly from the definition of contour integration and Lemma 5.4 above. □

Definition 5.6 (Convergence uniformly on compacts [Palka1991, Sec. VII.1.2])**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n \colon A \to \mathbb{C}$ on $A \subset \mathbb{C}$, and let $f \colon A \to \mathbb{C}$ also be a such function. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ **converges uniformly on compacts** (UOC) to f if for every compact subset $K \subset A$ the restrictions $f_n|_K \colon K \to \mathbb{C}$ converge uniformly on K to $f|_K \colon K \to \mathbb{C}$. We then write

$$f_n \xrightarrow{\text{UOC}} f \qquad \text{as } n \to \infty.$$

(This notion is also called by the alternative names **locally uniform convergence** and **normal convergence**.)

Lemma 5.7 (UOC limit of analytic functions [Palka1991, Thm VII.3.1])**.** *Suppose that functions $f_1, f_2, \ldots \colon U \to \mathbb{C}$ are analytic functions on an open set $U \subset \mathbb{C}$ and the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compacts to a function f. Then f is analytic on U. Moreover, for any $k \in \mathbb{N}$, the sequence $(f_n^{(k)})_{n\in\mathbb{N}}$ of kth derivatives converges uniformly on compacts to $f^{(k)}$.*

Proof. ... □

5.2 Complex series

Definition 5.8 (Complex series [Palka1991, Sec. VII.2.1])**.** Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ be complex numbers. For $N \in \mathbb{N}$, define the Nth **partial sum** of these as

$$S_N \;=\; \sum_{n=1}^{N} z_n \;=\; z_1 + z_2 + \cdots + z_N\,.$$

We say that the series $\sum_{n=1}^{\infty} z_n$ **converges** if the sequence $(S_N)_{N\in\mathbb{N}}$ of partial sums has a limit, and we then denote

$$\sum_{n=1}^{\infty} z_n = \lim_{N\to\infty} \sum_{n=1}^{N} z_n\,.$$

(Obvious modifications to the above definition are made if the terms' indexing starts from $n = 0$ or some other index, and the notation is correspondingly changed to, e.g., $\sum_{n=0}^{\infty}$.)

Lemma 5.9 (Terms of a convergent series tend to zero)**.** *If a complex series $\sum_{n=1}^{\infty} z_n$ converges, then we have*

$$\lim_{n\to\infty} z_n = 0.$$

Proof. ... □

Lemma 5.10 (Geometric series)**.** *The geometric series*

$$\sum_{n=0}^{\infty} z^n \;=\; 1 + z + z^2 + z^3 + \cdots,$$

with ratio $z \in \mathbb{C}$ converges if and only if $|z| < 1$. In that case its sum is

$$\sum_{n=0}^{\infty} z^n \;=\; \frac{1}{1-z}.$$

Proof. ... □

Definition 5.11 (Absolute convergence of a complex series)**.** A complex series $\sum_{n=1}^{\infty} z_n$ is said to **converge absolutely** if the series of absolute values $\sum_{n=1}^{\infty} |z_n|$ converges.

Lemma 5.12 (Absolute convergence implies convergence)**.** *If a complex series converges absolutely, then it converges.*

Proof. ... □

Lemma 5.13 (D'Alembert's ratio test)**.** *Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series such that the limit*

$$r = \lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|}$$

exists. Then:

(i) If $r < 1$, then the series $\sum_{n=1}^{\infty} z_n$ converges absolutely.

(ii) If $r > 1$, then the series $\sum_{n=1}^{\infty} z_n$ does not converge.

Proof. ... □

5.3 Series of functions

Definition 5.14 (Series of functions [Palka1991, Sec. VII.2.2])**.** Let $f_1, f_2, f_3, \ldots$ be complex-valued functions on a set A. For $N \in \mathbb{N}$, define their Nth **partial sum** function $F_N : A \to \mathbb{C}$ by

$$F_N(z) \;=\; \sum_{n=1}^{N} f_n(z) \;=\; f_1(z) + \cdots + f_N(z)\,.$$

We say that the function series $\sum_{n=1}^{\infty} f_n$ **converges pointwise** if the sequence $\big(F_N(z)\big)_{N\in\mathbb{N}}$ of partial sums has a limit at every $z \in A$. We say that the function series $\sum_{n=1}^{\infty} f_n$ **converges uniformly** on A if the sequence $(F_N)_{N\in\mathbb{N}}$ of partial sum functions converges uniformly on A. We say that the function series $\sum_{n=1}^{\infty} f_n$ **converges uniformly on compacts** if the sequence $(F_N)_{N\in\mathbb{N}}$ of partial sum functions converges uniformly on compacts.

The limit function is then denoted by $\sum_{n=1}^{\infty} f_n$.

(Obvious modifications to the above are made if the terms' indexing starts from $n = 0$ or some other index, and the notation is correspondingly changed to, e.g., $\sum_{n=0}^{\infty}$.)

Lemma 5.15 (Weierstrass M-test [Palka1991, Thm VII.2.2])**.** *Suppose that $M_1, M_2, \ldots \geq 0$ are nonnegative numbers such that the series $\sum_{n=1}^{\infty} M_n$ converges. Suppose also that for each $n \in \mathbb{N}$, $f_n : X \to \mathbb{C}$ is a function on X such that $|f_n(x)| \leq M_n$ for all $x \in X$. Then the series $\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly on X.*

Proof. ... □

Lemma 5.16 (Series of analytic functions [Palka1991, Thm VII.3.2])**.** *Suppose that functions $f_1, f_2, \ldots : U \to \mathbb{C}$ are analytic functions on an open set $U \subset \mathbb{C}$ such that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on compacts to a function $f : U \to \mathbb{C}$. Then f is analytic on U. Moreover, for any $k \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} f_n^{(k)}$ of kth derivatives converges uniformly on compacts to $f^{(k)}$.*

Proof. ... □

5.4 Power series

Definition 5.17 (Power series [Palka1991, Sec. VII.3.3])**.** Let $z_0 \in \mathbb{C}$ be a point in the complex plane and let $a_0, a_1, a_2 \ldots \in \mathbb{C}$ be coefficients. A function series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

is called a **power series** centered at z_0.

Lemma 5.18 (Abel's theorem)**.** *If a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges at $z = w \in \mathbb{C}$, then it converges absolutely for all $z \in \mathbb{C}$ such that $|z - z_0| < |w - z_0|$.

Proof. ... □

Corollary 5.19 (Abel's theorem in the contrapositive)**.** *If a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

does not converge at $z = w \in \mathbb{C}$, then it does not converge at any $z \in \mathbb{C}$ such that $|z - z_0| > |w - z_0|$.

Proof. ... □

Definition 5.20 (Radius of convergence)**.** The **radius of convergence** of a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is defined as

$$R := \sup \left\{ |z - z_0| \;\middle|\; \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges} \right\}.$$

From Lemma 5.18 and Corollary 5.19 it follows that the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for all $z \in \mathbb{C}$ such that $|z - z_0| < R$ and diverges for all $z \in \mathbb{C}$ such that $|z - z_0| > R$. The disk $\mathcal{B}(z_0; R)$ is called the **disk of convergence** of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

(If $R = +\infty$, we interpret $\mathcal{B}(z_0; R) = \mathbb{C}$.)

Lemma 5.21 (D'Alembert's ratio test for the radius of convergence)**.** *Suppose that for the coefficients of a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

the limit

$$\rho = \lim_{n \to \infty} \frac{|a_n|}{|a_{n+1}|}$$

exists. Then the radius of convergence R of the power series is $R = \rho$.

Proof. ... □

Theorem 5.22 (Hadamard's formula for the radius of convergence [Palka1991, Thm VII.3.3])**.** *Let $z_0 \in \mathbb{C}$ be a point in the complex plane and let $a_0, a_1, a_2 \ldots \in \mathbb{C}$ be coefficients. The radius of convergence of a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is given by the formula

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

with the conventions $\frac{1}{+\infty} = 0$ and $\frac{1}{0} = +\infty$.

Proof. ... □

Lemma 5.23 (Analyticity of power series [Palka1991, Thm VII.3.3])**.** *Let $z_0 \in \mathbb{C}$ be a point in the complex plane and let $a_0, a_1, a_2 \ldots \in \mathbb{C}$ be coefficients. Suppose that the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has radius of convergence $R > 0$. Then it defines an analytic function f on the disk $\mathcal{B}(z_0; R)$. The derivative of f is given by the power series

$$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1}.$$

Moreover, the coefficients a_k are related to the kth derivatives of f at z_0 through the formula

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Proof. ... □

Lemma 5.24 (Uniqueness of power series representation)**.** *Suppose that two power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ converge in a disk $\mathcal{B}(z_0; r)$ of radius $r > 0$ and represent the same function*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n \qquad \text{for } z \in \mathcal{B}(z_0; r).$$

Then their coefficients must be equal: $a_n = b_n$ for all n.

Proof. ... □

5.5 Taylor series and local representation of analytic functions

Theorem 5.25 (Taylor series of analytic functions [Palka1991, Thm VII.3.4])**.** *Suppose that $f\colon U \to \mathbb{C}$ is an analytic function on an open set $U \subset \mathbb{C}$ which contains a disk $\mathcal{B}(z_0; r) \subset U$. Then the function f can be represented in $\mathcal{B}(z_0; r)$ as a power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Moreover, this is the unique power series centered at z_0 that representats f in a neighborhood of z_0.

Proof. ... □

Theorem 5.26 (Equivalent characterizations of analyticity)**.** *Let $f\colon U \to \mathbb{C}$ be a continuous function on an open set $U \subset \mathbb{C}$. Then the following are equivalent:*

- *f is analytic on U;*
- *for any $z \in U$ there exists a neighborhood of z in which f has a primitive;*
- *for any $z \in U$ there exists a neighborhood of z in which f can be represented as a convergent power series.*

Proof. ... □

Lemma 5.27 (No vanishing of all derivatives at a point [Palka1991, Thm VIII.1.1])**.** *Suppose that $f\colon \mathcal{D} \to \mathbb{C}$ is an analytic function on a connected open set $\mathcal{D} \subset \mathbb{C}$. If there exists a point $z_0 \in \mathcal{D}$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$, then f is a constant function.*

Proof. ... □

Theorem 5.28 (Factor theorem for analytic functions [Palka1991, Thm VIII.1.2])**.** *Suppose that $f\colon \mathcal{D} \to \mathbb{C}$ is a non-constant analytic function on a connected open set $\mathcal{D} \subset \mathbb{C}$, and $z_0 \in \mathcal{D}$ is a point where $f(z_0) = 0$. Then f can be uniquely represented as*

$$f(z) = (z - z_0)^m \, g(z) \qquad \text{for } z \in \mathcal{D},$$

where $m \in \mathbb{N}$ and $g\colon \mathcal{D} \to \mathbb{C}$ is an analytic function such that $g(z_0) \neq 0$.

Proof. ... □

Corollary 5.29 (Local representation of analytic functions [Palka1991, Cor VIII.1.3])**.** *Suppose that $f\colon \mathcal{D} \to \mathbb{C}$ is a non-constant analytic function on a connected open set $\mathcal{D} \subset \mathbb{C}$. Then for any $z_0 \in \mathcal{D}$, we can write f uniquely in the form*

$$f(z) = f(z_0) + (z - z_0)^m \, g(z) \qquad \text{for } z \in \mathcal{D},$$

where $m \in \mathbb{N}$ and $g\colon \mathcal{D} \to \mathbb{C}$ is an analytic function such that $g(z_0) \neq 0$.

Proof. Apply Theorem 5.28 to the function $z \mapsto f(z) - f(z_0)$. □

Theorem 5.30 (L'Hospital's rule for analytic functions [Palka1991, Thm VIII.1.4])**.** *Let f and g be functions that are analytic in a neighborhood of z_0 such that $f(z_0) = 0$ and $g(z_0) = 0$. Then we have*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)},$$

understood in the sense that either both limits exist and are equal to each other, or else neither limit exists.

Proof. ... □

Theorem 5.31 (Discrete mapping theorem [Palka1991, Thm VIII.1.5])**.** *Suppose that $f\colon \mathcal{D} \to \mathbb{C}$ is a non-constant analytic function on a connected open set $\mathcal{D} \subset \mathbb{C}$. Then the set of zeros of f is discrete, i.e., for every $z_0 \in \mathcal{D}$ such that $f(z_0) = 0$, there exists a $r > 0$ such that $f(z) \neq 0$ for all $z \in \mathcal{B}(z_0; r) \setminus \{z_0\}$.*

Proof. ... □

Corollary 5.32 (Principle of analytic continuation [Palka1991, Cor VIII.1.6])**.** *Let $f, g\colon \mathcal{D} \to \mathbb{C}$ be two analytic functions on a connected open set $\mathcal{D} \subset \mathbb{C}$. If $f(z) = g(z)$ for all z in some subset of $\mathcal{D}$ which has an accumulation point in $\mathcal{D}$, then we have $f(z) = g(z)$ for all $z \in \mathcal{D}$.*

Proof. ... □

5.6 Laurent series

Definition 5.33 (Doubly infinite series [Palka1991, Sec. VII.2.1])**.** A doubly infinite series of complex numbers is a series of the form

$$\sum_{n=-\infty}^{\infty} z_n = \cdots + z_{-2} + z_{-1} + z_0 + z_1 + z_2 + \cdots,$$

where $\dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots \in \mathbb{C}$. We say that such a series **converges** to $s \in \mathbb{C}$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m_+ \geq N$ and $m_- \leq -N$ we have

$$\Big| \sum_{n=m_-}^{m_+} z_n - s \Big| < \varepsilon.$$

Lemma 5.34 (Convergence of doubly infinite series [Palka1991, Lem VII.2.1])**.** *A doubly infinite series*

$$\sum_{n=-\infty}^{\infty} z_n = \cdots + z_{-2} + z_{-1} + z_0 + z_1 + z_2 + \cdots,$$

of complex numbers converges if and only if both the series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ converge.

Proof. ... □

Definition 5.35 (Laurent series [Palka1991, Sec. VII.3.4]). A **Laurent series** centered at $z_0 \in \mathbb{C}$ is a doubly infinite series of functions of the form

$$
\begin{aligned}
z \mapsto & \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \\
&= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots .
\end{aligned}
$$

Lemma 5.36 (Annulus of convergence of Laurent power series [Palka1991, Thm VII.3.5]). *Consider a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.$$

Denote

$$\rho_- = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}, \qquad \rho_+ = \Big(\limsup_{n\to\infty} \sqrt[n]{|a_n|} \Big)^{-1}.$$

Then the series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ *converges for all* z *is the annulus*

$$\mathcal{A}_{\rho_-,\rho_+}(z_0) := \Big\{ z \in \mathbb{C} \;\Big|\; \rho_- < |z-z_0| < \rho_+ \Big\}.$$

Moreover, the convergence is uniform on compact subsets of $\mathcal{A}_{\rho_-,\rho_+}(z_0)$*, and the series defines an analytic function* $f(z)$ *on the annulus* $\mathcal{A}_{\rho_-,\rho_+}(z_0)$*.*

Proof. ... □

Theorem 5.37 (Laurent series for analytic functions [Palka1991, Thm VII.3.6]). *Suppose that* $f\colon U \to \mathbb{C}$ *is an analytic function on an open set* $U \subset \mathbb{C}$ *which contains an annulus*

$$\mathcal{A}_{r_1,r_2}(z_0) = \Big\{ z \in \mathbb{C} \;\Big|\; r_1 < |z-z_0| < r_2 \Big\}$$

for some $z_0 \in \mathbb{C}$ *and* $0 \le r_1 < r_2$*. Then the function* f *can be uniquely represented in* $\mathcal{A}_{r_1,r_2}(z_0)$ *as a series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, (z-z_0)^n,$$

where the coefficients a_n*, for* $n \in \mathbb{Z}$*, are given by*

$$a_n = \frac{1}{2\pi\mathrm{i}} \oint_{\partial\mathcal{B}(z_0;r)} \frac{f(z)}{(z-z_0)^{n+1}} \,\mathrm{d}z \qquad \text{for any } r \in (r_1, r_2).$$

Proof. ... □

Chapter 6

Isolated singularities and residues

6.1 The extended complex plane

Definition 6.1 (The Riemann sphere)**.** The **extended complex plane** is the set

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} ,$$

where ∞ is a symbol added to $\mathbb{C}$ to represent a single point at infinity. The set $\hat{\mathbb{C}}$ is given a topology in such a way that open sets in $\mathbb{C}$ remain open in $\hat{\mathbb{C}}$, and sets of the form $\{z \in \mathbb{C} \mid |z| > M\}$ for $M > 0$ form a neighborhood basis at ∞.

(This topology makes $\hat{\mathbb{C}}$ homeomorphic to the 2-dimensional sphere in three-dimensional space, and $\hat{\mathbb{C}}$ is also called the **Riemann sphere**.)

For example a function $f \colon U \to \mathbb{C}$ has limit $\lim_{z\to z_0} f(z) = \infty$ at z_0 if for any $M > 0$ there exists a $\delta > 0$ such that $|f(z)| > M$ whenever $0 < |z - z_0| < \delta$.

6.2 Isolated singularities of analytic functions

Definition 6.2 (Isolated singularity [Palka1991, Sec. VIII.2.1])**.** Let $f : U \to \mathbb{C}$ be an analytic function on an open set $U \subset \mathbb{C}$. We say that f has an **isolated singularity** at $z_0 \in \mathbb{C}$ if $\mathcal{B}(z_0; r) \setminus \{z_0\} \subset U$ for some $r > 0$ but $z_0 \notin U$.

Definition 6.3 (Classification of isolated singularities [Palka1991, Sec. VIII.2.1])**.** Let $z_0 \in \mathbb{C}$ be an isolated singularity of an analytic function $f : U \to \mathbb{C}$. Let $r > 0$ be such that $\mathcal{B}(z_0; r) \setminus \{z_0\} \subset U$, so that by Theorem 5.37 f can be represented in $\mathcal{B}(z_0; r) \setminus \{z_0\}$ uniquely as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Depending on the coefficients a_n of negative indices $n < 0$, we distinguish three types of singularities:

- f has a **removable singularity** at z_0 if $a_n = 0$ for all $n < 0$;
- f has a **pole of order** $m \in \mathbb{N}$ at z_0 if $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$;
- f has an **essential singularity** at z_0 if $a_n \neq 0$ for infinitely many $n < 0$.

Definition 6.4 (Residue at an isolated singularity [Palka1991, Sec. VIII.2.1]). Let $z_0 \in \mathbb{C}$ be an isolated singularity of an analytic function $f : U \to \mathbb{C}$. Let $r > 0$ be such that $\mathcal{B}(z_0; r) \setminus \{z_0\} \subset U$, so that by Theorem 5.37 f can be represented in $\mathcal{B}(z_0; r) \setminus \{z_0\}$ uniquely as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The coefficient a_{-1} is called the **residue** of f at z_0, and is denoted $\mathrm{Res}_{z_0}(f) = a_{-1} \in \mathbb{C}$.

Theorem 6.5 (Removable singularity characterization [Palka1991, Thm VIII.2.1 and Thm VIII.2.2]). *Let $z_0 \in \mathbb{C}$ be an isolated singularity of an analytic function $f : U \to \mathbb{C}$. Then the following conditions are equivalent:*

(R-1) *The singularity of f at z_0 is removable (i.e., all negative index Laurent series coefficients of f expanded near z_0 vanish).*

(R-2) *There exists an analytic function $\tilde{f} \colon U \cup \{z_0\} \to \mathbb{C}$ such that $f(z) = \tilde{f}(z)$ for all $z \in U$.*

(R-3) *The limit $\lim_{z \to z_0} f(z)$ exists in $\mathbb{C}$.*

(R-4) *The function f is bounded in some punctured disk $\mathcal{B}(z_0; r) \setminus \{z_0\}$ with $r > 0$.*

Proof. ... □

Theorem 6.6 (Characterization of poles [Palka1991, Thm VIII.2.3 and Thm VIII.2.4]). *Let $z_0 \in \mathbb{C}$ be an isolated singularity of an analytic function $f : U \to \mathbb{C}$. Then the following conditions are equivalent:*

(P-1) *The singularity of f at z_0 is a pole (i.e., finitely many Laurent series coefficients of f near z_0 are nonzero).*

(P-2) *There exists an $m \in \mathbb{N} = \{1, 2, \ldots\}$ such that $z \mapsto (z - z_0)^m f(z)$ has a removable singularity and a nonzero limit as $z \to z_0$.*

(P-3) *The function f has the limit $\lim_{z \to z_0} f(z) = \infty$ at z_0.*

Proof. ... □

Theorem 6.7 (Characterization of essential singularities [Palka1991, Thm VIII.2.6 and Thm VIII.2.7]). *Let $z_0 \in \mathbb{C}$ be an isolated singularity of an analytic function $f : U \to \mathbb{C}$. Then the following conditions are equivalent:*

(E-1) *The singularity of f at z_0 is essential (i.e., infinitely many Laurent series coefficients of f near z_0 are nonzero).*

(E-2) *For any small $\delta > 0$, the image $f\big[\mathcal{B}(z_0; \delta) \setminus \{z_0\}\big]$ is dense in $\mathbb{C}$.*

(E-3) *The limit $\lim_{z \to z_0} f(z)$ does not exist in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.*

Proof. ... □

6.3 The residue theorem

Theorem 6.8 (Residue theorem [Palka1991, Thm VIII.3.1])**.** *Let $U \subset \mathbb{C}$ be an open set and γ a contractible closed contour in U. Let $f : U \setminus S \to \mathbb{C}$ be an analytic function with isolated singularities at a countable set $S \subset U$ of points. Then*

$$\oint_\gamma f(z)\, dz = 2\pi\mathfrak{i} \sum_{w \in S} \mathfrak{n}_w(\gamma) \operatorname{Res}_w(f).$$

Proof. ... $\square$

Corollary 6.9 (Residue theorem for Jordan contours [Palka1991, Cor VIII.3.2])**.** *Let $U \subset \mathbb{C}$ be an open set and $S \subset U$ a discrete subset of it. Let $\mathcal{D}$ be a Jordan domain such that $\overline{\mathcal{D}} \subset U$ and $\partial\mathcal{D} \cap S = \emptyset$. Let γ be a closed contour traversing the boundary $\partial\mathcal{D}$ of the Jordan domain in the positive orientation. Let $f : U \setminus S \to \mathbb{C}$ be an analytic function with isolated singularities at the points of S. Then*

$$\oint_\gamma f(z)\, dz = 2\pi\mathfrak{i} \sum_{w \in S \cap \mathcal{D}} \operatorname{Res}_w(f).$$

Proof. ... $\square$

Appendix A

Topological preliminaries

A.1 Metrics and related concepts

Definition A.1 (Metric)**.** A **metric** on a set X is a function $\mathsf{d}\colon X \times X \to [0, \infty)$ such that for all $p_1, p_2, p_3 \in X$ we have

$$\begin{aligned} &\mathsf{d}(p_1, p_3) \le \mathsf{d}(p_1, p_2) + \mathsf{d}(p_2, p_3) && \text{(triangle inequality)} \\ &\mathsf{d}(p_1, p_2) = \mathsf{d}(p_2, p_1) && \text{(symmetricity)} \\ &\mathsf{d}(p_1, p_2) = 0 \text{ if and only if } p_1 = p_2. && \text{(separation of points)} \end{aligned}$$

The set X equipped with the metric d on it is called a **metric space**.

Lemma A.2 (Metric in the complex plane)**.** *The formula*

$$\mathsf{d}(z, w) = |z - w| \qquad \textit{for } z, w \in \mathbb{C}$$

defines a metric on the complex plane $\mathbb{C}$.

(Thus $\mathbb{C}$ becomes a metric space. Also any subset of $\mathbb{C}$, in particular $\mathbb{R} \subset \mathbb{C}$, becomes a metric space when equipped with the metric given by the above formula restricted to the subset.)

Proof. ... □

Definition A.3 (Ball (disk))**.** Let X be a metric space with metric $\mathsf{d}\colon X \times X \to [0, \infty)$. Let $p_0 \in X$ be a point and let $r > 0$.

The set

$$\mathcal{B}(p_0; r) = \left\{ p \in X \;\middle|\; \mathsf{d}(p, p_0) < r \right\}$$

is called an **open ball** in X, centered at p_0, and with radius r.

The set

$$\overline{\mathcal{B}}(p_0; r) = \left\{ p \in X \;\middle|\; \mathsf{d}(p, p_0) \le r \right\}$$

is called a **closed ball** in X, centered at p_0, and with radius r.

(In the case of the complex plane $\mathbb{C}$, the term **disk** is often used instead of the general metric space theory term **ball**.)

Definition A.4 (Interior point)**.** Let X be a metric space, and $A \subset X$ a subset. A point $p \in A$ is said to be an **interior point** of A if for some $r > 0$ we have $\mathcal{B}(p;r) \subset A$.

Definition A.5 (Exterior point)**.** Let X be a metric space, and $A \subset X$ a subset. A point $p \in X \setminus A$ is said to be an **exterior point** of A if for some $r > 0$ we have $\mathcal{B}(p;r) \subset X \setminus A$.

(It is easy to see that the exterior points of A are exactly the interior points)

Definition A.6 (Boundary)**.** Let X be a metric space, and $A \subset X$ a subset. A point $p \in X$ is said to be a **boundary point** of A if for all $r > 0$ we have that $\mathcal{B}(p;r)$ contains points of A and $X \setminus A$ (i.e. $\mathcal{B}(p;r) \cap A \neq \emptyset$ and $\mathcal{B}(p;r) \setminus A \neq \emptyset$).

The set of all boundary points of A is denoted ∂A and called the **boundary** of A.

(It is easy to see that the boundary $\partial A \subset X$ is exactly the set of points of X which are neither interior nor exterior points of A.)

Definition A.7 (Open set)**.** Let X be a metric space. A subset $U \subset X$ is said to be an **open set** if each point $p \in U$ is an interior point of U.

Definition A.8 (Closed set)**.** Let X be a metric space. A subset $F \subset X$ is said to be a **closed set** if the complement $X \setminus F \subset X$ is an open set.

(Equivalently, each point $p \in X \setminus F$ in the complement of F is an exterior point of F.)

Definition A.9 (Boundedness)**.** Let X be a metric space. with metric $\mathsf{d} \colon X \times X \to [0, \infty)$.

A subset $A \subset X$ is **bounded** if there exists a number $M > 0$ such that $\mathsf{d}(p, q) \leq M$ for all $p, q \in A$. (If X is nonempty, an equivalent definition would be that A is bounded if it is a subset of some ball in X.)

A function $f \colon Z \to X$ with values in a metric space X is **bounded** if the set $f[Z] \subset X$ of its values is a bounded subset of X.

(In the case $X = \mathbb{C}$ we have the following further characterizations: A subset $A \subset \mathbb{C}$ is bounded if and only if there exists an $R > 0$ such that $|z| \leq R$ for all $z \in A$. A function $f \colon Z \to \mathbb{C}$ is bounded if and only if there exists an $R > 0$ such that $|f(z)| \leq R$ for all $z \in Z$.)

A.2 Limits

Definition A.10 (Limit)**.** Let X be a metric space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in X. We say that the sequence $(x_n)_{n \in \mathbb{N}}$ **converges** to a **limit** $x \in X$ if for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in \mathcal{B}(x;\varepsilon)$ (i.e., $\mathsf{d}(x_n, x) < \varepsilon$). We then denote

$$\lim_{n \to \infty} x_n = x.$$

(It is straightforward to check that the limit is unique if it exists.)

Let then X and Y be metric spaces, with respective metrics d_X and d_Y, and let $f \colon X \to Y$ be a function. We say that the function f has a **limit** $y \in Y$ **at a point** $p_0 \in X$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $p \in \mathcal{B}(p_0;\delta) \setminus \{p_0\}$ we have $f(p) \in \mathcal{B}(y;\varepsilon)$. We then denote

$$\lim_{p \to p_0} f(p) = y.$$

(It is straightforward to check that the limit is unique if it exists.)

(Equivalently, written in terms of distances, $\lim_{p \to p_0} f(p) = y$ means that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that we have $\mathsf{d}_Y(f(p), y) < \varepsilon$ whenever $0 < \mathsf{d}_X(p, p_0) < \delta$.)

Lemma A.11 (Limits in the complex plane)**.** *For a sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers we have*

$$\lim_{n\to\infty} z_n = z$$

if and only if

$$\lim_{n\to\infty} \mathfrak{Re}(z_n) = \mathfrak{Re}(z) \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{Im}(z_n) = \mathfrak{Im}(z).$$

Let X be a metric space, let $f\colon X \to \mathbb{C}$ a complex-valued function on X, and let $p_0 \in X$ be a point. Then we have

$$\lim_{p\to p_0} f(p) = z$$

if and only if

$$\lim_{p\to p_0} \mathfrak{Re}\big(f(p)\big) = \mathfrak{Re}(z) \quad \text{and} \quad \lim_{p\to p_0} \mathfrak{Im}\big(f(p)\big) = \mathfrak{Im}(z).$$

Proof. ... □

Lemma A.12 (Operations with complex limits)**.** *Let $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be complex number sequences converging to limits*

$$\lim_{n\to\infty} z_n = z \quad \text{and} \quad \lim_{n\to\infty} w_n = w.$$

Then we have

$$\lim_{n\to\infty} (z_n + w_n) \;=\; z + w, \quad \lim_{n\to\infty} (z_n w_n) \;=\; zw, \quad \lim_{n\to\infty} \frac{z_n}{w_n} \;=\; \frac{z}{w} \;\; \text{if } w \neq 0.$$

Let X be a metric space, let $p_0 \in X$ be a point, and let $f, g\colon X \to \mathbb{C}$ be two complex-valued functions on X such that

$$\lim_{p\to p_0} f(p) = z \quad \text{and} \quad \lim_{p\to p_0} g(p) = w.$$

Then we have

$$\lim_{p\to p_0} \big(f(p) + g(p)\big) \;=\; z + w, \quad \lim_{p\to p_0} \big(f(p)\, g(p)\big) \;=\; zw, \quad \lim_{p\to p_0} \frac{f(p)}{g(p)} \;=\; \frac{z}{w} \;\; \text{if } w \neq 0.$$

Proof. The arguments are similar to the proofs given in **MS-C1541 Metric Spaces** for the real-valued cases. □

Definition A.13 (Cauchy sequence)**.** ...

Lemma A.14 (Every real Cauchy sequence converges)**.** *If a real number sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, then it converges to a limit $\lim_{n\to\infty} x_n \in \mathbb{R}$.*

*(This property is known as **completeness** of the metric space $\mathbb{R}$.)*

Proof. See **MS-C1541 Metric Spaces**. □

Lemma A.15 (Every complex Cauchy sequence converges)**.** *If a complex number sequence $(z_n)_{n\in\mathbb{N}}$ is Cauchy, then it converges to a limit $\lim_{n\to\infty} z_n \in \mathbb{C}$.*

*(This property is known as **completeness** of the metric space $\mathbb{C}$.)*

Proof. See **MS-C1541 Metric Spaces**.

(Idea: This follows from Lemma A.14 by considering real and imaginary parts separately and picking a subsequence of a subsequence.) □

A.3 Continuity

Definition A.16 (Continuity)**.** Let X and Y be metric spaces. A function $f\colon X \to Y$ is said to be **continuous at a point** $p_0 \in X$ if $\lim_{p \to p_0} f(p) = f(p_0)$.

(Equivalently, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $p \in \mathcal{B}(p_0; \delta)$ we have $f(p) \in \mathcal{B}(f(p_0); \varepsilon)$.)

A function $f\colon X \to Y$ is said to be **continuous** if it is continuous at every point $p_0 \in X$.

Lemma A.17 (Continuity of complex-valued functions)**.** *Let X be a metric space, and let $f\colon X \to \mathbb{C}$ be a complex-valued function on X. Then f is continuous at $p_0 \in X$ if and only if its real and imaginary parts $p \mapsto \mathfrak{Re}(f(p))$ and $p \mapsto \mathfrak{Im}(f(p))$ are continuous at p_0.*

Proof. ... ☐

Corollary A.18 (Continuity of coordinate projections)**.** *The coordinate projections*

$$\begin{aligned} \mathfrak{Re}\colon\ & \mathbb{C} \to \mathbb{R} \\ & z \mapsto \mathfrak{Re}(z) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \mathfrak{Im}\colon\ & \mathbb{C} \to \mathbb{R} \\ & z \mapsto \mathfrak{Im}(z) \end{aligned}$$

are continuous functions.

Proof. ... ☐

Lemma A.19 (Operations with continuous complex-valued functions)**.** *Let X be a metric space, let $p_0 \in X$ be a point, and let $f, g\colon X \to \mathbb{C}$ be two complex-valued functions on X which are continuous at p_0. Then also the functions*

$$p \mapsto f(p) + g(p) \quad \textit{and} \quad p \mapsto f(p)\, g(p)$$

are continuous at p_0.

If moreover $g(p_0) \neq 0$, then also the function $p \mapsto \frac{f(p)}{g(p)}$ is continuous at p_0.

Proof. ... ☐

Lemma A.20 (Continuity characterization)**.** *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a function. Then the following are equivalent:*

- *f is a continuous function;*
- *for every open set $V \subset Y$, the preimage $f^{-1}[V] = \{x \in X \,|\, f(x) \in V\}$ is an open set in X;*
- *for every closed set $A \subset Y$, the preimage $f^{-1}[A] = \{x \in X \,|\, f(x) \in A\}$ is a closed set in X.*

Proof. See **MS-C1541 Metric Spaces**. ☐

Lemma A.21 (Composition of continuous functions)**.** *Let X, Y, and Z be metric spaces, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. If f is continuous at $x_0 \in X$ and g is continuous at $f(x_0) \in Y$, then the composition $g \circ f\colon X \to Z$ is continuous at x_0.*

(The composition $g \circ f$ is defined by the formula $(g \circ f)(x) = g(f(x))$.)

Proof. ... ☐

Corollary A.22 (Real multivariate polynomials are continuous)**.** *Let $N \in \mathbb{N}$ be a natural number, and let $c_{n,m} \in \mathbb{R}$ be real numbers for $n, m \in \{0, 1, \ldots, N\}$. Then the function $p \colon \mathbb{C} \to \mathbb{R}$ defined by*

$$p(x + \mathrm{i}y) = \sum_{m=0}^{N} \sum_{n=0}^{N} c_{m,n}\, x^m\, y^n$$

is continuous.

Proof. See **MS-C1541 Metric Spaces**. □

Definition A.23 (Uniform continuity)**.** Let X and Y be metric spaces. A function $f \colon X \to Y$ is **uniformly continuous** if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $p_0 \in X$ and $p \in \mathcal{B}(p_0; \delta)$ we have $f(p) \in \mathcal{B}(f(p_0); \varepsilon)$.

Lemma A.24 (Uniform continuity implies continuity)**.** *If a function $f \colon X \to Y$ is uniformly continuous, then it is continuous.*

Proof. See **MS-C1541 Metric Spaces**.

(The easy proof is also a good exercise.) □

A.4 Connectedness and path-connectedness

Definition A.25 (Connectedness)**.** A set $A \subset X$ in a metric space X is **disconnected** if there exists a continuous surjective function $f \colon A \to \{0, 1\}$ onto the two-element discrete set $\{0, 1\}$. Otherwise A is **connected**; then every continuous function $A \to \{0, 1\}$ must be either constant 0 or constant 1.

(The usual definition in topology textbooks reads slightly differently, but it is equivalent to the one we chose here by Lemma A.20.)

Definition A.26 (Path-connectedness)**.** A set $A \subset X$ in a metric space X is **path connected** if for any two points $p, q \in X$ there exists a continuous function $\gamma \colon [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$ (a parametrized path in X starting from p and ending at q).

Lemma A.27 (Path-connectedness implies connectedness)**.** *If a metric space X is path-connected, then it is connected.*

Proof. See **MS-C1541 Metric Spaces**. □

Lemma A.28 (Open connected sets are path-connected)**.** *Suppose that $U \subset \mathbb{C}$ is an open subset of the complex plane. Then U is connected if and only if it is path-connected.*

Proof. See **MS-C1541 Metric Spaces**. □

A.5 Compactness

Definition A.29 (Compactness)**.** Let X be a metric space. A subset $K \subset X$ is **compact** if every sequence $(x_n)_{n \in \mathbb{N}}$ of points $x_n \in K$ has a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ which converges to a limit $\lim_{k \to \infty} x_{n_k} \in K$ in the set K.

Theorem A.30 (Bolzano-Weierstrass theorem)**.** *A subset $B \subset \mathbb{R}$ of the real line is compact if an only if it is closed and bounded.*

A subset $A \subset \mathbb{C}$ of the complex plane is compact if an only if it is closed and bounded.

Proof. See **MS-C1541 Metric Spaces**. □

Theorem A.31 (Boundedness of continuous functions on compacts)**.** *Suppose that X is compact. Then every continuous function $f\colon X \to \mathbb{R}$ is bounded.*

Proof. ... □

Lemma A.32 (On a compact domain continuity implies uniform continuity)**.** *If X is compact and a function $f\colon X \to Y$ is continuous, then it is uniformly continuous.*

Proof. See **MS-C1541 Metric Spaces**. □

Lemma A.33 (Continuous bijection from a compact domain is a homeomorphism)**.** *Let X and Y be metric spaces and assume that X is compact. Then for any continuous bijection $f\colon X \to Y$, also the inverse $f^{-1}\colon Y \to X$ is continuous.*

Proof. See **MS-C1541 Metric Spaces**. □

Theorem A.34 (Cantor's intersection theorem [Palka1991, Thm II.4.5])**.** *Let X be a metric space. Suppose that $K_1, K_2, K_3, \ldots$ are nonempty compact subsets of X nested so that $K_1 \supset K_2 \supset K_3 \supset \cdots$. Then the intersection $\bigcap_{n=1}^{\infty} K_n$ is nonempty.*

Proof. See **MS-C1541 Metric Spaces**. □

A.6 Simple connectedness

Definition A.35 (Path homotopy for closed paths)**.** Let X be a metric space and $\gamma_0\colon [a,b] \to X$ and $\gamma_0\colon [a,b] \to X$ two closed paths in X. If there exists a continuous function (called a **homotopy**)

$$\Gamma\colon [0,1] \times [a,b] \to X$$

such that

$$\Gamma(0,t) = \gamma_0(t) \quad \text{and} \quad \Gamma(1,t) = \gamma_1(t) \qquad \text{for all } t \in [a,b]$$

and

$$\Gamma(s,a) = \Gamma(s,b) \qquad \text{for all } s \in [0,1],$$

then we say that the closed paths γ_0 and γ_1 are **homotopic**.

Definition A.36 (Contractible path)**.** Let X be a metric space. A closed path $\gamma\colon [a,b] \to X$ is called **contractible** if it is homotopic to a constant path.

Definition A.37 (Simple connectedness)**.** A metric space is said to be **simply connected** if every closed path $\gamma\colon [a,b] \to X$ in X is contractible.

Appendix B

Preliminaries from calculus

B.1 Differentiability

Definition B.1 (Real differentiability)**.** Let $m, n \in \mathbb{N}$, and let $f\colon U \to \mathbb{R}^m$ be a function defined on a subset $U \subset \mathbb{R}^n$. A linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be a **differential** of f at $p_0 \in U$ if

$$f(p) = f(p_0) + L(p - p_0) + E(p - p_0)$$

where the error term E is small near p_0 in the sense that

$$\lim_{p \to p_0} \frac{\|E(p - p_0)\|}{\|p - p_0\|} = 0.$$

We say that f is **differentiable** at p_0 if such a linear map L exists.

It is easy to check that the differential L of f at p_0 is unique if p_0 is an interior point of U; we then denote it by $L = \mathrm{d}f(p_0)$.

Lemma B.2 (Differentiability implies continuity)**.** *If a function $f\colon U \to \mathbb{R}^m$ defined on a subset $U \subset \mathbb{R}^n$ is differentiable at $p_0 \in U$, then it is continuous at p_0.*

Proof. ... □

Lemma B.3 (Jacobian matrix of the differential)**.** *If a function $f\colon U \to \mathbb{R}^m$ defined on a subset $U \subset \mathbb{R}^n$ is differentiable at an interior point p_0 of U, then it has all first order partial derivatives at p_0, and the matrix representation of the differential $\mathrm{d}f(p_0)$ in the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ is*

$$\mathrm{d}f(p_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p_0) & \cdots & \frac{\partial f_1}{\partial x_n}(p_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p_0) & \cdots & \frac{\partial f_m}{\partial x_n}(p_0) \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $f_1, \dots, f_m \colon U \to \mathbb{R}$ denote component functions of f.

Proof. ... □

Lemma B.4 (Vanishing partial derivatives implies locally constant)**.** *Suppose that $f\colon U \to \mathbb{R}^m$ is a function defined on an open and connected subset $U \subset \mathbb{R}^n$ of $\mathbb{R}^n$ whose first order partial derivatives exist and are zero at all points of U. Then f is a constant function.*

Proof. ... □

B.2 Riemann integral

For the purposes of this course, it suffices to know the Riemann integral. (Those who already know Lebesgue integration theory can substitute that more general notion of integral everywhere.)

Definition B.5 (Riemann integral)**.** ...

Lemma B.6 (Riemann integrability of continuous functions)**.** *Any continuous function* $f\colon [a,b] \to \mathbb{R}$ *is Riemann integrable on* $[a,b]$*.*

Proof. See **MS-C1541 Metric Spaces**. □

B.3 Trigonometry

Lemma B.7 (Trigonometric angle sum identities)**.** *Let* $\alpha, \beta \in \mathbb{R}$*. Then we have*

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\,\cos(\beta) - \sin(\alpha)\,\sin(\beta)\\ \sin(\alpha+\beta) &= \cos(\alpha)\,\sin(\beta) + \sin(\alpha)\,\cos(\beta).\end{aligned}$$

Proof. ... □

B.4 Supremum, infimum, limit superior, and limit inferior

Definition B.8 (Supremum)**.** The **supremum**, or the **least upper bound**, of a set $A \subset \mathbb{R}$ is the smallest real number s such that $a \leq s$ for all $a \in A$, and is denoted by $s = \sup A$.

By the completeness axiom of real numbers, every nonempty set ($A \neq \emptyset$) of real numbers which is bounded from above (for some $u \in \mathbb{R}$ we have $a \leq u$ for all $a \in A$) has a supremum $\sup A \in \mathbb{R}$. We adopt the notational conventions that $\sup \emptyset = -\infty$, and that $\sup A = +\infty$ if A is not bounded from above.

For convenience, we also adopt some flexibility in the notation: for example the supremum of values of a real-valued function on a set D is denoted by

$$\sup_{x\in D} f(x) \;:=\; \sup\left\{ f(x) \;\middle|\; x \in D \right\}$$

and the supremum of values in the tail of a real-number sequence (x_n) starting from index m is denoted by

$$\sup_{n\geq m} x_n \;:=\; \sup\left\{ x_n \;\middle|\; m \geq n \right\}.$$

Definition B.9 (Infimum)**.** The **infimum**, or the **greatest lower bound**, of a set $A \subset \mathbb{R}$ is the greatest real number i such that $a \geq i$ for all $a \in A$, and is denoted by $i = \inf A$.

By the completeness axiom of real numbers, every nonempty set ($A \neq \emptyset$) of real numbers which is bounded from below (for some $\ell \in \mathbb{R}$ we have $a \geq \ell$ for all $a \in A$) has an infimum $\inf A \in \mathbb{R}$. We adopt the notational conventions that $\inf \emptyset = +\infty$, and that $\inf A = -\infty$ if A is not bounded from below.

For convenience, we also adopt some flexibility in the notation for infimums of function values or sequence values, similarly as with supremums.

Definition B.10 (Limit superior)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then the **limit superior** of the sequence is defined as

$$\limsup_{n\to\infty} x_n \;:=\; \lim_{m\to\infty}\Big(\sup_{n\geq m} x_n\Big).$$

With the following conventions, the limit superior of a sequence always exists as either a real number or one of the symbols $\pm\infty$. If the sequence is not bounded from above, then by conventions regarding the supremum, we have $\sup_{n\geq m} x_n = +\infty$ for every m, so we correspondingly set $\limsup_{n\to\infty} x_n = +\infty$. Otherwise the sequence $(\sup_{n\geq m} x_n)_{m\in\mathbb{N}}$ is a decreasing sequence of real numbers, so either it is bounded from below and converges to $\lim_{m\to\infty}\big(\sup_{n\geq m} x_n\big) = \inf_{m\in\mathbb{N}}\big(\sup_{n\geq m} x_n\big) \in \mathbb{R}$, or it is not bounded from below and we set $\limsup_{n\to\infty} x_n = \inf_{m\in\mathbb{N}}\big(\sup_{n\geq m} x_n\big) = -\infty$.

Definition B.11 (Limit inferior)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then the **limit inferior** of the sequence is defined as

$$\liminf_{n\to\infty} x_n \;:=\; \lim_{m\to\infty}\Big(\inf_{n\geq m} x_n\Big).$$

With the following conventions, the limit inferior of a sequence always exists as either a real number or one of the symbols $\pm\infty$. If the sequence is not bounded from below, then by conventions regarding the infimum, we have $\inf_{n\geq m} x_n = -\infty$ for every m, so we correspondingly set $\liminf_{n\to\infty} x_n = -\infty$. Otherwise the sequence $(\inf_{n\geq m} x_n)_{m\in\mathbb{N}}$ is an increasing sequence of real numbers, so either it is bounded from above and converges to $\lim_{m\to\infty}\big(\inf_{n\geq m} x_n\big) = \sup_{m\in\mathbb{N}}\big(\inf_{n\geq m} x_n\big) \in \mathbb{R}$, or it is not bounded from above and we set $\liminf_{n\to\infty} x_n = \sup_{m\in\mathbb{N}}\big(\inf_{n\geq m} x_n\big) = +\infty$.

Lemma B.12 (Limit with limsup and liminf)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and let $x \in \mathbb{R}$. Then the following are equivalent:*

- *The limit $\lim_{n\to\infty} x_n$ exists and equals x.*
- *We have both $\limsup_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} x_n = x$.*

(With the usual conventions of $\pm\infty$ as possible limits of real-number sequences, the above equivalence of conditions also extends to the cases $x = \pm\infty$.)

Proof. ... $\square$