Virasoro Project

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Chapter 1

Introduction

Under construction.

Chapter 2

Lie algebra cohomology in degree two

Let $\mathbb{K}$ be a field and let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Fix also a vector space $\mathfrak{a}$ over $\mathbb{K}$, (interpreted, when necessary, as an abelian Lie algebra, i.e., all Lie brackets in $\mathfrak{a}$ are taken to be zero).

Definition 1 (Lie algebra 1-cocycle)**.** A **1-cocycle** of the Lie algebra $\mathfrak{g}$ with coefficients in the vector space $\mathfrak{a}$ is a linear map

$$\beta\colon \mathfrak{g} \to \mathfrak{a}.$$

The set of all such 1-cocycles is denoted by $C^1(\mathfrak{g}, \mathfrak{a})$.

Definition 2 (Lie algebra 2-cocycle)**.** A **2-cocycle** of the Lie algebra $\mathfrak{g}$ with coefficients in the vector space $\mathfrak{a}$ is a bilinear map

$$\gamma\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$$

such that for all $X \in \mathfrak{g}$ we have the antisymmetry condition

$$\gamma(X, X) = 0 \tag{2.1}$$

and for all $X, Y, Z \in \mathfrak{g}$ we have the Leibnitz rule

$$\gamma(X, [Y, Z]) = \gamma([X, Y], Z) + \gamma(Y, [X, Z]). \tag{2.2}$$

The set of all such 2-cocycles is denoted by $C^2(\mathfrak{g}, \mathfrak{a})$.

Lemma 3 (Skew-symmetry of 2-cocycles)**.** *For any $\gamma \in C^2(\mathfrak{g}, \mathfrak{a})$ and any $X, Y \in \mathfrak{g}$, we have the skew-symmetry property*

$$\gamma(X, Y) = -\gamma(Y, X).$$

Proof. The Leibnitz rule (4.1) applied to $X + Y$ gives

$$\begin{aligned} 0 &= \gamma(X + Y, X + Y) \\ &= \gamma(X, X) + \gamma(X, Y) + \gamma(Y, X) + \gamma(Y, Y) \end{aligned}$$

by bilinearity of γ. The first and the last terms in the last expression vanish by antisymmetry (2.1), and the asserted skew-symmetry equation follows. $\square$

Lemma 4 (Lie algebra 1-cocycles form a vector space)**.** *The set $C^1(\mathfrak{g},\mathfrak{a})$ of 1-cocycles of $\mathfrak{g}$ with coefficients in $\mathfrak{a}$ forms a vector space over $\mathbb{K}$.*

Proof. By definition, $C^1(\mathfrak{g},\mathfrak{a})$ is the space of linear maps $\mathfrak{g}\to\mathfrak{a}$, and such linear maps form a vector space. $\square$

Lemma 5 (Lie algebra 2-cocycles form a vector space)**.** *The set $C^2(\mathfrak{g},\mathfrak{a})$ of 2-cocycles of $\mathfrak{g}$ with coefficients in $\mathfrak{a}$ forms a vector space over $\mathbb{K}$.*

Proof. The conditions defining $C^2(\mathfrak{g},\mathfrak{a})$ are linear, so this is staightforward. $\square$

Definition 6 (Lie algebra 2-coboundary)**.** Given a 1-cocycle $\beta\in C^1(\mathfrak{g},\mathfrak{a})$, we define the **coboundary** $\partial\beta$ of β to be the bilinear map

$$\partial\beta\colon\ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{a}$$

given by

$$\partial\beta(X,Y)=\beta([X,Y]).$$

We then have $\partial\beta\in C^2(\mathfrak{g},\mathfrak{a})$. The mapping $\partial\colon C^1(\mathfrak{g},\mathfrak{a})\to C^2(\mathfrak{g},\mathfrak{a})$ is linear. Its range is denoted $B^2(\mathfrak{g},\mathfrak{a})\subset C^2(\mathfrak{g},\mathfrak{a})$ and called the set of **2-coboundaries** of the Lie algebra $\mathfrak{g}$ with coefficients in $\mathfrak{a}$.

Definition 7 (Lie algebra 2-cohomology)**.** The vector space

$$H^2(\mathfrak{g},\mathfrak{a}):=C^2(\mathfrak{g},\mathfrak{a})\,/\,B^2(\mathfrak{g},\mathfrak{a})$$

is called the **Lie algebra cohomology in degree 2** of $\mathfrak{g}$ with coefficients in $\mathfrak{a}$.

Lemma 8 (Cohomology of abelian Lie algebras)**.** *If $\mathfrak{g}$ is abelian, i.e., $[\mathfrak{g},\mathfrak{g}]=0$, then the canonical projection*

$$C^2(\mathfrak{g},\mathfrak{a})\to H^2(\mathfrak{g},\mathfrak{a})$$

is a linear isomorphism.

Proof. The projection is surjective by construction, so it suffices to show that it is also injective. The kernel of the projection is $B^2(\mathfrak{g},\mathfrak{a})=\mathrm{Im}\,\partial$. In view of Definition 6, abelianity of $\mathfrak{g}$ implies $\partial\beta=0$ for any $\beta\in C^1(\mathfrak{g},\mathfrak{a})$. Therefore $B^2(\mathfrak{g},\mathfrak{a})=0$, and the kernel of the projection is trivial, so the projection is indeed injective. $\square$

Chapter 3

Central extensions of Lie algebras

3.1 Central extensions of Lie algebras

Definition 9 (Lie algebra extension)**.** An **extension** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ by a Lie algebra $\mathfrak{a}$ is a Lie algebra together with a pair of two Lie algebra homomorphisms $\iota\colon \mathfrak{a} \longrightarrow \mathfrak{h}$ and $\pi\colon \mathfrak{h} \longrightarrow \mathfrak{g}$ which form a short exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

i.e., such that ι is injective, π is surjective, and $\mathrm{Im}(\iota) = \mathrm{Ker}(\pi)$.

Definition 10 (Lie algebra central extension)**.** A **central extension** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ is a Lie algebra extension

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

such that $\mathrm{Im}(\iota)$ is contained in the centre of $\mathfrak{h}$, i.e., $[\iota(A), W] = 0$ for all $A \in \mathfrak{a}$, $W \in \mathfrak{h}$.

3.2 Central extensions determined by 2-cocycles

Definition 11 (Central extension determined by a cocycle)**.** Given a Lie algebra 2-cocycle $\gamma \in C^2(\mathfrak{g}, \mathfrak{a})$, on the vector space

$$\mathfrak{h}_\gamma = \mathfrak{g} \oplus \mathfrak{a}$$

define a bracket by

$$[(X, A), (Y, B)]_\gamma := ([X, Y]_{\mathfrak{g}}, \gamma(X, Y)).$$

Then $\mathfrak{h}_\gamma$ becomes a Lie algebra with the Lie bracket $[\cdot, \cdot]_\gamma$.

Lemma 12 (Central extension determined by cohomologous cocycles)**.** *Let $\gamma_1, \gamma_2 \in C^2(\mathfrak{g}, \mathfrak{a})$ be two Lie algebra 2-cocycles and $\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}$ the central extensions corresponding to these according to Definition 11. If the two 2-cocycles differ by a coboundary, $\gamma_2 - \gamma_1 = \partial\beta$ with some $\beta \in C^1(\mathfrak{g}, \mathfrak{a})$, then the mapping $\mathfrak{h}_{\gamma_1} \to \mathfrak{h}_{\gamma_2}$ given by*

$$(X, A) \mapsto (X, A + \beta(X))$$

is an isomophism of Lie algebras $\mathfrak{h}_{\gamma_1} \cong \mathfrak{h}_{\gamma_2}$.

Proof. The mapping $\phi_\beta\colon \mathfrak{h}_1 \to \mathfrak{h}_2$ given by

$$\phi_\beta((X,A)) := (X, A+\beta(X))$$

is clearly linear. It is also bijective, since the similarly defined mapping $\phi_{-\beta}\colon \mathfrak{h}_2 \to \mathfrak{h}_1$, $\phi_{-\beta}((X,A)) := (X, A-\beta(X))$, is a two-sided inverse to ϕ_β. So it remains to verify that this bijective linear map $\phi_\beta\colon \mathfrak{h}_1 \to \mathfrak{h}_2$ is in fact a homomorphism Lie algebras.

Let $(X,A),(Y,B) \in \mathfrak{g}\oplus\mathfrak{a} = \mathfrak{h}_{\gamma_1}$. The bracket in $\mathfrak{h}_{\gamma_1}$ of these is, by definition,

$$[(X,A),(Y,B)]_{\gamma_1} := ([X,Y]_\mathfrak{g}, \gamma_1(X,Y)).$$

Applying the mapping ϕ_β to this, we get

$$\phi_\beta\Big([(X,A),(Y,B)]_{\gamma_1}\Big) = ([X,Y]_\mathfrak{g}, \gamma_1(X,Y) + \beta([X,Y]_\mathfrak{g})).$$

On the other hand the Lie bracket in $\mathfrak{h}_2$ of the images is

$$\begin{aligned}
&[\phi_\beta(((X,A)), \phi_\beta((Y,B))]_{\gamma_2}\\
&= [(X, A+\beta(X)),(Y, B+\beta(Y))]_{\gamma_2}\\
&= \Big([X,Y]_\mathfrak{g}, \gamma_2(X,Y)\Big)\\
&= \Big([X,Y]_\mathfrak{g}, \gamma_1(X,Y) + \beta([X,Y]_\mathfrak{g})\Big).
\end{aligned}$$

From the equality of these two expressions we see that ϕ_β indeed is also a Lie algebra homomorphism. ☐

Lemma 13 (Central extension determined by a cocycle is a central extension)**.** *Given a Lie algebra 2-cocycle $\gamma \in C^2(\mathfrak{g},\mathfrak{a})$, consider the Lie algebra $\mathfrak{h}_\gamma = \mathfrak{g}\oplus\mathfrak{a}$ as in Definition 11. With the inclusion $\iota\colon \mathfrak{a} \to \mathfrak{g}\oplus\mathfrak{a}$ in the second direct summand and the projection $\pi\colon \mathfrak{g}\oplus\mathfrak{a} \to \mathfrak{g}$ to the first direct summand, the Lie algebra $\mathfrak{h}_\gamma = \mathfrak{g}\oplus\mathfrak{a}$ becomes a central extension of $\mathfrak{g}$ by $\mathfrak{a}$, i.e., we have the short exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h}_\gamma \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0.$$

Proof. Clearly

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{h}_\gamma \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is an exact sequence of vector spaces, and it is straightforward to check with Definition 11 that ι and π are Lie algebra homomorphisms. ☐

Theorem 14. *Every cohomology class in $H^2(\mathfrak{g},\mathfrak{a})$ determines a well-defined isomorphism class of central extensions of the Lie algebra $\mathfrak{g}$ by $\mathfrak{a}$ by the rule that the class $[\gamma] \in H^2(\mathfrak{g},\mathfrak{a})$ of a cocycle $\gamma \in C^2(\mathfrak{g},\mathfrak{a})$ corresponds to the isomorphism class of $\mathfrak{h}_\gamma$ (Definition 11).*

Proof. This follows from Lemmas 13 and 12. ☐

Chapter 4

Witt algebra and its 2-cohomology

4.1 Definition of the Witt algebra

Definition 15 (Witt algebra)**.** Let $\mathbb{K}$ be a field (or a commutative ring). The **Witt algebra** over $\mathbb{K}$ is the $\mathbb{K}$-vector space $\mathfrak{witt}$ (or free $\mathbb{K}$-module) with basis $(\ell_n)_{n\in\mathbb{Z}}$ and a $\mathbb{K}$-bilinear bracket $\mathfrak{witt} \times \mathfrak{witt} \to \mathfrak{witt}$ given on basis elements by

$$[\ell_n, \ell_m] = (n - m)\, \ell_{n+m}.$$

With some assumptions on $\mathbb{K}$, the Witt algebra $\mathfrak{witt}$ with the above bracket is an ∞-dimensional Lie algebra over $\mathbb{K}$.

Lemma 16 (Witt algebra is a Lie algebra)**.** *If $\mathbb{K}$ is a field of characteristic zero, then $\mathfrak{witt}$ is a Lie algebra over $\mathbb{K}$.*

(In the case when $\mathbb{K}$ is a commutative ring, the $\mathfrak{witt}$ is also a Lie algebra assuming the $\mathbb{K}$ has characteristic zero and that for $c \in \mathbb{K}$ and $X \in \mathfrak{witt}$ we have $c \cdot X = 0$ only if either $c = 0$ or $X = 0$.)

Proof. By construction, the bracket in Definition 15 is bilinear. It is antisymmetric on the basis vectors ℓ_n, $n \in \mathbb{Z}$, so by bilinearity the bracket is antisymmetric. It remains to check that the bracket satisfies Leibnitz rule (or the Jacobi identity), i.e., that for any $X, Y, X \in \mathfrak{witt}$ we have

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

This formula is trilinear in X, Y, Z, so it suffices to verify it on basis vectors $X = \ell_n$, $Y = \ell_m$, $Z = \ell_k$. Calculating, with Definition 15, we have

$$[\ell_n, [\ell_m, \ell_k]] = [\ell_n, (m - k)\ell_{m+k}] = (m - k)(n - (m + k))\, \ell_{n+m+k}$$

and

$$\begin{aligned} &[[\ell_n, \ell_m], \ell_k] + [\ell_m, [\ell_n, \ell_k]] \\ =\ &[(n - m)\ell_{n+m}, \ell_k] + [\ell_m, (n - k)\ell_{n+k}]] \\ =\ &(n - m)(n + m - k)\, \ell_{n+m+k} + (n - k)(m - (n + k))\, \ell_{m+n+k}. \end{aligned}$$

Noting that

$$(n-m)(n+m-k)+(n-k)(m-(n+k))=(m-k)(n-(m+k)),$$

the Leibniz rule follows. $\square$

4.2 Virasoro cocycle

In this section we assume that $\mathbb{K}$ is a field of characteristic zero and $\mathfrak{witt}$ is the Witt algebra over $\mathbb{K}$ as in Definition 15.

Definition 17 (Virasoro cocycle)**.** The bilinear map $\gamma_{\mathfrak{vir}}\colon \mathfrak{witt}\times\mathfrak{witt}\to\mathbb{K}$ given on basis elements of $\mathfrak{witt}$ by

$$\gamma_{\mathfrak{vir}}(\ell_n,\ell_m)=\frac{n^3-n}{12}\,\delta_{n+m,0}$$

is called the **Virasoro cocycle**.

Lemma 18 (The Virasoro cocycle is a 2-cocycle)**.** *The Virasoro cocycle is a 2-cocycle, $\gamma_{\mathfrak{vir}}\in C^2(\mathfrak{witt},\mathbb{K})$.*

Proof. By the construction if Definition 17, $\gamma_{\mathfrak{vir}}\colon \mathfrak{witt}\times\mathfrak{witt}\to\mathbb{K}$ is bilinear. It's antisymmetry on basis elements of the Witt algebra is easily checked, so $\gamma_{\mathfrak{vir}}$ is antisymmetric. It remains to prove the Leibniz rule for $\gamma_{\mathfrak{vir}}$, i.e., that for $X,Y,X\in\mathfrak{witt}$, we have

$$\gamma_{\mathfrak{vir}}(X,[Y,Z])=\gamma_{\mathfrak{vir}}([X,Y],Z)+\gamma_{\mathfrak{vir}}(Y,[X,Z]). \tag{4.1}$$

This formula is trilinear in X,Y,Z, so it suffices to verify it for basis vectors $X=\ell_n$, $Y=\ell_m$, $Z=\ell_k$. We calculate

$$\begin{aligned}\gamma_{\mathfrak{vir}}(\ell_n,[\ell_m,\ell_k])&=\gamma_{\mathfrak{vir}}(\ell_n,(m-k)\ell_{m+k}) &(4.2)\\ &=(m-k)\,\frac{n^3-n}{12}\delta_{n+m+k,0}. &(4.3)\end{aligned}$$

and

$$\begin{aligned}&\gamma_{\mathfrak{vir}}([\ell_n,\ell_m],\ell_k)+\gamma_{\mathfrak{vir}}(\ell_m,[\ell_n,\ell_k]) &(4.4)\\ &=\gamma_{\mathfrak{vir}}((n-m)\ell_{n+m},\ell_k)+\gamma_{\mathfrak{vir}}(\ell_m,(n-k)\ell_{n+k}) &(4.5)\\ &=(n-m)\,\frac{(n+m)^3-(n+m)}{12}\delta_{n+m+k,0}+(n-k)\,\frac{m^3-m}{12}\delta_{n+m+k,0}. &(4.6)\end{aligned}$$

Both of the above results are nonzero only if $k=-(n+m)$, in which case $m-k=2m+n$ and $n-k=2n+m$, so it suffices to note that

$$(2m+n)\,(n^3-n)=(n-m)\,((n+m)^3-(n+m))+(2n+m)(m^3-m)$$

to verify the Leibniz rule for $\gamma_{\mathfrak{vir}}$. $\square$

Lemma 19 (The Virasoro cocyle is nontrivial)**.** *The cohomology class $[\gamma_{\mathfrak{vir}}]\in H^2(\mathfrak{witt},\mathbb{K})$ of the Virasoro cocycle is nonzero.*

Proof. Assume, by way of contradiction, that $\gamma_{\mathfrak{vir}} \in B^2(\mathfrak{witt}, \mathbb{K})$, i.e., that $\gamma_{\mathfrak{vir}} = \partial\beta$ for some $\beta \in C^1(\mathfrak{witt}, \mathbb{K})$. Then, in particular, for every $n \in \mathbb{Z}$ we would have

$$\gamma_{\mathfrak{vir}}(\ell_n, \ell_{-n}) = \beta([\ell_n, \ell_{-n}]) = 2n\,\beta(\ell_0).$$

By Definition 17, this would imply

$$\frac{n^3 - n}{12} = 2n\,\beta(\ell_0)$$

for all $n \in \mathbb{Z}$. Considering for example $n = 3$ and $n = 6$, we then get

$$2 = 6\,\beta(\ell_0) \qquad \text{and} \qquad \frac{35}{2} = 12\,\beta(\ell_0),$$

which obviously yield a contradiction. □

4.3 Witt algebra 2-cohomology

Lemma 20 (Witt algebra 2-cocycle condition for basis)**.** *For any Witt algebra 2-cocycle $\gamma \in C^2(\mathfrak{witt}, \mathbb{K})$ with coefficients in $\mathbb{K}$, we have*

$$(m-k)\,\gamma(\ell_n, \ell_{m+k}) + (k-n)\,\gamma(\ell_m, \ell_{n+k}) + (n-m)\,\gamma(\ell_k, \ell_{n+m}) \;=\; 0$$

for all $n, m, k \in \mathbb{Z}$.

Proof. Direct calculation, using Definitions 15 and 2. □

Lemma 21 (Witt algebra 2-cocycle support assuming normalization)**.** *Let $\gamma \in C^2(\mathfrak{witt}, \mathbb{K})$ be a Witt algebra 2-cocycle such that $\gamma(\ell_0, \ell_n) = 0$ for all $n \in \mathbb{Z}$. Then for any $n, m \in \mathbb{Z}$ with $n + m \neq 0$, we have*

$$\gamma(\ell_n, \ell_m) = 0.$$

Proof. Apply Lemma 20 with $k = 0$. The last term vanishes, and by skew-symmetry of γ, the first two terms simplify to yield

$$(m+n)\,\gamma(\ell_n, \ell_m) = 0,$$

which, assuming $n + m \neq 0$, yields the asserted equation $\gamma(\ell_n, \ell_m) = 0$. □

Lemma 22 (Normalization of Witt algebra 2-cocycles)**.** *For any 2-cocycle $\gamma \in C^2(\mathfrak{witt}, \mathbb{K})$, there exists a coboundary $\partial\beta$ with $\beta \in C^1(\mathfrak{witt}, \mathbb{K})$ such that*

$$\gamma + \partial\beta \;=\; r \cdot \gamma_{\mathfrak{vir}}$$

for some scalar $r \in \mathbb{K}$.

Proof. Let $\gamma \in C^2(\mathfrak{witt}, \mathbb{K})$ be a Witt algebra 2-cocycle. Define a Witt algebra 1-cocycle $\beta \in C^1(\mathfrak{witt}, \mathbb{K})$ by linear extension of

$$\beta(\ell_n) = \begin{cases} -\frac{1}{2}\gamma(\ell_1, \ell_{-1}) & \text{if } n = 0 \\ \frac{1}{n}\gamma(\ell_0, \ell_n) & \text{if } n \neq 0. \end{cases}$$

For any $n \neq 0$, we calculate

$$\begin{aligned}(\gamma + \partial\beta)(\ell_0, \ell_n) &= \gamma(\ell_0, \ell_n) + \beta([\ell_0, \ell_n]) \\ &= \gamma(\ell_0, \ell_n) - n\,\beta(\ell_n) \\ &= \gamma(\ell_0, \ell_n) - n\,\frac{1}{n}\gamma(\ell_0, \ell_n) = 0.\end{aligned}$$

This property and Lemma 21 imply that

$$(\gamma + \partial\beta)(\ell_0, \ell_n) = 0$$

whenever $n + m \neq 0$.

We will show the asserted equation with

$$r = 2\,(\gamma + \partial\beta)(\ell_2, \ell_{-2}).$$

By comparison with the Virasoro cocycle $\gamma_{\mathfrak{vir}}$ of Definition 17, and using skew-symmetry, it remains to show that for any $n \in \mathbb{N}$ we have

$$(\gamma + \partial\beta)(\ell_n, \ell_{-n}) = r\,\frac{n^3 - n}{12}.$$

The case $n = 0$ is a direct consequence of antisymmetry. The case $n = 1$ follows using the definition of β and the calculation

$$\begin{aligned}(\gamma + \partial\beta)(\ell_1, \ell_{-1}) &= \gamma(\ell_1, \ell_{-1}) + \beta([\ell_1, \ell_{-1}]) \\ &= \gamma(\ell_1, \ell_{-1}) + 2\,\beta(\ell_0) \\ &= \gamma(\ell_1, \ell_{-1}) - 2\,\frac{1}{2}\gamma(\ell_1, \ell_{-1}) = 0.\end{aligned}$$

The case $n = 2$ follows directly by the choice of r. We prove the equality in the cases $n \geq 3$ by induction on n. Assume the equation for smaller values of n. Apply Lemma 20 to $\gamma + \partial\beta$ with $m = 1 - n$ and $k = -1$ to get

$$\begin{aligned}0 &= (2-n)\,(\gamma + \partial\beta)(\ell_n, \ell_{-n}) + (-1-n)\,(\gamma + \partial\beta)(\ell_{1-n}, \ell_{n-1}) + (2n-1)\,(\gamma + \partial\beta)(\ell_1, \ell_{-1}) \\ &= (2-n)\,(\gamma + \partial\beta)(\ell_n, \ell_{-n}) - (-1-n)\,r\frac{(n-1)^3 - (n-1)}{12} \\ &= (2-n)\,(\gamma + \partial\beta)(\ell_n, \ell_{-n}) + \frac{r}{12}(n+1)n(n-1)(n-2).\end{aligned}$$

where in the seciond step we used the induction hypothesis. Since $2 - n \neq 0$, this can be solved for

$$(\gamma + \partial\beta)(\ell_n, \ell_{-n}) = -\frac{r}{12}\frac{(n+1)n(n-1)(n-2)}{2-n} = r\frac{n^3 - n}{12},$$

completing the induction step. □

Lemma 23 (Witt algebra 2-cohomology is spanned by the Virasoro cocycle)**.** *The Lie algebra 2-cohomology $H^2(\mathfrak{witt}, \mathbb{K})$ of the Witt algebra $\mathfrak{witt}$ with coefficients in $\mathbb{K}$ is one-dimensional and spanned by the class of the Virasoro cocycle $\gamma_{\mathfrak{vir}}$,*

$$H^2(\mathfrak{witt}, \mathbb{K}) \;=\; \mathbb{K} \cdot [\gamma_{\mathfrak{vir}}].$$

Proof. This follows directly from Lemmas 22 and 19. □

Chapter 5

Virasoro algebra

Definition 24 (Virasoro algebra)**.** Let $\mathbb{K}$ be a field of characteristic zero. The **Virasoro algebra $\mathfrak{vir}$** is the Lie algebra over $\mathbb{K}$ obtained as the central extension of the Witt algebra $\mathfrak{witt}$ corresponding to the Virasoro cocycle $\gamma_{\mathfrak{vir}} \in C^2(\mathfrak{witt}, \mathbb{K})$.

From the definition of the Virasoro algebra and the Virasoro cocycle, Definition 24 and 17, we directly obtain that $\mathfrak{vir}$ has a basis of the following form.

Definition 25 (The standard basis of the Virasoro algebra)**.** The Virasoro algebra $\mathfrak{vir}$ has a basis consisting of $(L_n)_{n\in\mathbb{Z}}$ and C, with Lie brackets determined by the following

$$[L_n, L_m] = (n-m)\, L_{n+m} + \delta_{n+m,0} \frac{n^3 - n}{12}\, C,$$

$$[C, L_n] = 0, \qquad [C, C] = 0,$$

for $n, m \in \mathbb{Z}$.

Chapter 6

Heisenberg algebra

In this section we assume that $\mathbb{K}$ is a field of characteristic zero.

Definition 26 (Heisenberg cocycle)**.** Let $\mathfrak{g}$ be the vector space with basis $(j_k)_{k\in\mathbb{Z}}$ over $\mathbb{K}$, considered as an abelian Lie algebra. The bilinear map $\gamma_{\mathfrak{hei}} \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ given on basis elements by

$$\gamma_{\mathfrak{hei}}(j_k, j_l) = k\,\delta_{k+l,0}$$

is a Lie algebra 2-cocycle, $\gamma_{\mathfrak{hei}} \in C^2(\mathfrak{g}, \mathbb{K})$. We call $\gamma_{\mathfrak{hei}}$ the **Heisenberg cocycle**.

Lemma 27 (The Heisenberg cocyle is nontrivial)**.** *The cohomology class* $[\gamma_{\mathfrak{hei}}] \in H^2(\mathfrak{g}, \mathbb{K})$ *of the Heisenberg cocycle is nonzero.*

Proof. ... □

Definition 28 (Heisenberg algebra)**.** Let $\mathbb{K}$ be a field of characteristic zero. The **Heisenberg algebra** $\mathfrak{hei}$ is the Lie algebra over $\mathbb{K}$ obtained as the central extension of the abelian Lie algebra $\mathfrak{g}$ with basis $(j_k)_{k\in\mathbb{Z}}$, corresponding to the Heisenberg cocycle $\gamma_{\mathfrak{hei}} \in C^2(\mathfrak{g}, \mathbb{K})$.

From the definition of the Heisenberg algebra and the Heisenberg cocycle, Definition 28 and 26, we directly obtain that $\mathfrak{hei}$ has a basis of the following form.

Definition 29 (The standard basis of the Heisenberg algebra)**.** The Heisenberg algebra $\mathfrak{hei}$ has a basis consisting of $(J_k)_{k\in\mathbb{Z}}$ and K, with Lie brackets determined by the following

$$[J_k, J_l] = k\,\delta_{k+l,0}\,K, \quad [K, J_k] = 0, \quad [K, K] = 0,$$

for $k, l \in \mathbb{Z}$.

Chapter 7

Sugawara construction

7.1 The basic bosonic Sugawara construction

Throughout this section, let $\mathbb{K}$ be a field of characteristic zero.

If a vector space V has a representation of the Heisenberg algebra on a vector space V, where the central element K (see Definition 29), acts as id_V, then the basis elements $(J_k)_{k\in\mathbb{Z}}$ (see Definition 29) are linear operators $\mathsf{J}_k : V \to V$ satisfying the commutation relations

$$\text{(HeiComm)} \qquad [\mathsf{J}_k, \mathsf{J}_l] \;=\; \mathsf{J}_k \circ \mathsf{J}_l - \mathsf{J}_l \circ \mathsf{J}_k \;=\; k\,\delta_{k+l,0}\,\mathrm{id}_V.$$

Below we will assume such operators being fixed, and satisfying furthermore the local truncation condition on V: for any fixed $v \in V$ we have $\mathsf{J}_k\, v = 0$ for $k \gg 0$, i.e.,

$$\text{(HeiTrunc)} \qquad \forall v \in V,\ \exists N,\ \forall k \geq N, \quad \mathsf{J}_k\, v = 0.$$

Definition 30 (Normal ordering)**.** For $k, l \in \mathbb{Z}$, we denote the normal ordered product of the operators J_k and J_l by

$$:\mathsf{J}_k\,\mathsf{J}_l: \;:=\; \begin{cases} \mathsf{J}_k \circ \mathsf{J}_l & \text{if } k \leq l \\ \mathsf{J}_l \circ \mathsf{J}_k & \text{if } k > l. \end{cases}$$

Lemma 31 (Alternative normal ordering)**.** *Suppose that $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ satisfy the commutation relations (HeiComm). Then for any $k, l \in \mathbb{Z}$ we have*

$$:\mathsf{J}_k\,\mathsf{J}_l: \;=\; \begin{cases} \mathsf{J}_k \circ \mathsf{J}_l & \text{if } k < 0 \\ \mathsf{J}_l \circ \mathsf{J}_k & \text{if } k \geq 0. \end{cases}$$

Proof. Straightforward using the commutation relations (HeiComm). □

Lemma 32 (Local truncation for normal ordered products)**.** *Suppose that $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ satisfy the local truncation condition (HeiTrunc). Then for any $v \in V$ there exists an N such that whenever $\max\{k, l\} \geq N$ we have $:\mathsf{J}_k\,\mathsf{J}_l: v = 0$.*

Proof. Fixing $v \in V$, the local truncation condition (HeiTrunc) gives the existence of an N such that $\mathsf{J}_k\, v = 0$ for $k \geq N$. It is then clear by inspection of Definition 30 that $:\mathsf{J}_k\,\mathsf{J}_l: v = 0$ when $\max\{k, l\} \geq N$. □

Lemma 33 (Local finite support for homogeneous normal ordered products)**.** *Suppose that* $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ *satisfy the local truncation condition (HeiTrunc). Then for any* $n \in \mathbb{Z}$ *and any* $v \in V$*, there are only finitely many* $k \in \mathbb{Z}$ *such that* $:\mathsf{J}_{n-k}\,\mathsf{J}_k: v \neq 0$*.*

Proof. Straightforward from Lemma 32. $\square$

Definition 34 (Sugawara operators)**.** Suppose that $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ satisfy the local truncation condition (HeiTrunc). Then for any $n \in \mathbb{Z}$, a linear operator

$$\mathsf{L}_n \colon V \to V$$

can be defined by the formula

$$\mathsf{L}_n\, v := \frac{1}{2} \sum_{k\in\mathbb{Z}} :\mathsf{J}_{n-k}\,\mathsf{J}_k: v \qquad \text{for } v \in V$$

(the sum only has finitely many terms by Lemma 33).

We call the operators $(\mathsf{L}_n)_{n\in\mathbb{Z}}$ the **Sugawara operators**.

Lemma 35 (Commutators of Sugawara operators as series)**.** *Suppose that* $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ *satisfy the local truncation condition (HeiTrunc), and suppose that* $\mathsf{A} \colon V \to V$ *is a linear operator. Then for any* $n \in \mathbb{Z}$*, the action of the commutator* $[\mathsf{L}_n, \mathsf{A}]$ *on any* $v \in V$ *is given by the series*

$$[\mathsf{L}_n, \mathsf{A}]\, v = \frac{1}{2} \sum_{k\in\mathbb{Z}} [:\mathsf{J}_{n-k}\,\mathsf{J}_k:, \mathsf{A}]\, v$$

where only finitely many of the terms are nonzero.

Proof. Write

$$\begin{aligned}
[\mathsf{L}_n, A]\, v &= \mathsf{L}_n\, A\, v - A\, \mathsf{L}_n\, v \\
&= \frac{1}{2} \sum_{k\in\mathbb{Z}} :\mathsf{J}_{n-k}\,\mathsf{J}_k: \mathsf{A}\, v - \frac{1}{2} \mathsf{A} \sum_{k\in\mathbb{Z}} :\mathsf{J}_{n-k}\,\mathsf{J}_k: v.
\end{aligned}$$

By Lemma 33, only finitely many of the terms in both sums are nonzero and they may be rearranged to the asserted form of sum of commutators. The resulting sum only has finitely many nonzero terms and is therefore well-defined. $\square$

Lemma 36 (Commutator of Sugawara operators with Heisenberg operators)**.** *Suppose that* $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ *satisfy the commutation relations (HeiComm) and the local truncation condition (HeiTrunc). Then for any* $n \in \mathbb{Z}$ *and* $k \in \mathbb{Z}$*, we have*

$$[\mathsf{L}_n, \mathsf{J}_k] \;=\; -k\, \mathsf{J}_{n+k}.$$

Proof. Calculation, using Lemma 35 and the commutator formula $[A, BC] = B[A, C] + [A, B]C$. $\square$

Lemma 37 (Commutator of Sugawara operators with normal ordered pairs)**.** *Suppose that* $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ *satisfy the commutation relations (HeiComm) and the local truncation condition (HeiTrunc). Then for any* $n \in \mathbb{Z}$ *and* $k, m \in \mathbb{Z}$*, we have*

$$\begin{aligned}
[\mathsf{L}_n, :\mathsf{J}_k\,\mathsf{J}_{m-k}:] = &\; -k :\mathsf{J}_{n+k}\,\mathsf{J}_{m-k}: - (m-k) :\mathsf{J}_k\,\mathsf{J}_{n+m-k}: \\
&+ (n+k)\, \delta_{n+m,0} \Big(\mathbb{1}_{-n\le k<0} - \mathbb{1}_{0\le k<-n} \Big)\, \mathrm{id}_V.
\end{aligned}$$

where $\mathbb{1}_{\text{condition}}$ *is defined as* 1 *if the condition is true and* 0 *otherwise.*

Proof. Calculation, using Lemmas 35, 31, and 36, the commutation relations (HeiComm), and the commutator formula $[A, BC] = B[A, C] + [A, B]C$ again. □

Lemma 38 (Auxiliary calculation)**.** *For any $n \in \mathbb{N}$, we have*

$$\sum_{l=0}^{n-1} (n-l)l = \frac{n^3 - n}{6}.$$

Proof. Calculation (with induction). □

Lemma 39 (Virasoro commutation relations for Sugawara operators)**.** *Suppose that $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ satisfy the commutation relations (HeiComm) and the local truncation condition (HeiTrunc). Then for any $n, m \in \mathbb{Z}$, we have*

$$[\mathsf{L}_n, \mathsf{L}_m] = (n-m)\, \mathsf{L}_{n+m} + \delta_{n+m,0} \frac{n^3 - n}{12} \, \mathrm{id}_V.$$

Proof. Calculation, using Lemmas 37 and 38, among other observations. □

Theorem 40 (Sugawara construction)**.** *Suppose that $(\mathsf{J}_k)_{k\in\mathbb{Z}}$ satisfy the commutation relations (HeiComm) and the local truncation condition (HeiTrunc). Then there exists a representation of the Virasoro algebra $\mathfrak{vir}$ with central charge $c = 1$ on V (i.e., the central element $C \in \mathfrak{vir}$ acts as $c \, \mathrm{id}_V$ with $c = 1$) where the basis elements L_n of $\mathfrak{vir}$ act by the Sugawara operators $(\mathsf{L}_n)_{n\in\mathbb{Z}}$.*

Proof. A direct consequence of the commutation relations in Lemma 39 and a comparison with the Lie brackets in the basis of Definition 25. □